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The Eigenvalues-Based Entropy and Spectrum of the Directed Cycles

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ABSTRACT

The directed cycles form a foundational structure within a network model. By analyzing the in-degree characteristic polynomial of three kinds of matrices of the directed cycles, the authors obtain the eigenvalues of the adjacency matrix $\boldsymbol{A}^{-}$, the Laplacian matrix $\boldsymbol{L}^{-}$, and the signless Laplacian matrix $\boldsymbol{Q}^{-}$. This study investigates the eigenvalues spectrum of these three types of matrices for directed cycles and introduces an eigenvalue-based entropy calculated from the real part of the eigenvalues. The computer simulation reveals interesting characteristics on the spectrum of the signless Laplacian. The concept of eigenvalue-based entropy holds promise for enhancing our understanding of graph neural networks and more applications of social networks.

KEYWORDS

Computer Simulation, Directed Cycles, Eigenvalues, Entropy, Signless Laplacian Matrix

1. INTRODUCTION

Both nature and society are complex systems. The complex structure serves as a model for studying problems related to complex network (Biggs, 2014; Brooks, 1993). A complex network is typically represented as a graph, with vertices and edges (Barnett, 1993; Chan et al., 1997). For instance, in technical networks, computers and routers are represented as vertices, while the physical connections between them are represented as edges (Ho & Dooren, 2005; Abreu, 2007; Godsil & Royle, 2001). Similarly, in a scientific collaboration network, scientists can be seen as vertices, and if two researchers co-author an article, the two vertices are connected by an edge (Anton & Rorres, 2014).

Spectrum and entropy have been popular topics in graph theory for a long time (Mowshowitz, 1968; Bapat, 2014). While there have been numerous results on undirected graphs (Cvetkovic et al.,1998), there is less research on directed graphs due to the asymmetry of the matrix. Directed cycles

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are common structures in directed graphs (Chung, 1997; Butler & Chung, 2013). Cyclic networks can be modeled using nearest neighbor coupling networks. This work discusses the spectrum and entropy of cyclic directed complex networks, calculates the eigenvalue-based entropy of cyclic networks, and verifies them in constructed nearest neighbor coupling networks through computer simulation.

This study contributes to a better understanding of graph neural networks (GCN) and more applications in social networks. The entropy of directed cyclic serves as an indicator of social computing (Bruna et al., 2014; Bronstein et al., 2017. During the process of applying real-life scenarios to game, research on cyclic networks is crucial for analyzing the level-up algorithms and some metaverse applications.

2. PRELIMINARY

Let $G=\left(V,E\right)$ be a directed graph, where $V=\left\{v_1,v_2,\cdots,v_n\right\}$ is the set of vertices and E is the set of ordered vertex pairs $\left(v_i,v_j\right)$, which are edges of the directed graph. The edge is also denoted by $v_i \to v_j$. The basic property of a vertex in a digraph is the degree of the vertex. The in-degree and out-degree of a vertex i are denoted by d_i^- and d_i^+, respectively. The in-degree sum of vertex v_i is denoted by $\sum_{v_j \to v_i} d_i^-$. The degree matrix D is defined as the diagonal $n \times n$ matrix, where each vertex's degree is at the position of the main diagonal, and the term outside the main diagonal is zero. Similarly, in a digraph, the in-degree and the out-degree matrix can be defined as D^- and D^+, respectively. Let the in-degree adjacency matrix of the digraph be denoted by A^-, in-degree Laplacian matrix be denoted by L^-, and in-degree signless Laplacian matrix be denoted by Q^- (Cvetkovic et al., 1998).

Definition 1. The in-degree adjacency matrix A^-, which has elements a_{ji}^- ($i,j \in \left\{1,2,\cdots,n\right\}$), is defined as:

$$a_{ji}^{-}=\begin{cases}0 & v_i=v_j \\ 1 & if\, there\, is\, an\, arc\, from\, v_j\, to\, v_i, or\, v_j \to v_i \\ 0 & others\end{cases}$$

For directed graphs, the adjacency matrix is asymmetric (Butler & Chung, 2013).

Definition 2. The in-degree Laplacian matrix L^-, which has elements l_{ji}^- ($i,j \in \left\{1,2,\cdots,n\right\}$), is defined as:

$$l_{ji}^{-}=\begin{cases}d_i^- & v_i=v_j \\ -1 & if\, there\, is\, an\, arc\, from\, v_j\, to\, v_i, or\, v_j \to v_i. \\ 0 & others\end{cases}$$

Definition 3. The in-degree Laplacian matrix Q^-, which has elements q_{ji}^- ($i,j \in \left\{1,2,\cdots,n\right\}$), is defined as:

$$q_{ji}^{\ -}=\begin{cases} d_i^- & v_i=v_j \\ 1 & if\, there\, is\, an\, arc\, from\, v_j\, to\, v_i, or\, v_j \to v_i. \\ 0 & others \end{cases}$$

Definition 4. Let the distribution of information be $s_1, s_2, \cdots, s_n$. The information entropy can be defined as:

$$I_S=-\sum_{i=1}^{n} s_i \log\left(s_i\right). \tag{1}$$

For the characteristic polynomial $f\left(x\right)=a_1x+a_2x^2+a_3x^3+\cdots+a_nx^n$, the distribution of the eigenvalues $\left\{p_1, p_2, \cdots, p_i, \cdots, p_n\right\}$ of the characteristic polynomial $f\left(x\right)$ is defined as:

$$p_i=\frac{\left|\lambda_i\right|}{\sum_{k=1}^{n}\left|\lambda_i\right|}, \tag{2}$$

where λ_i is eigenvalue of characteristic polynomial. Then the eigenvalue entropy of the characteristic polynomial is defined as:

$$I_P=-\sum_{i=1}^{n} p_i \log\left(p_i\right). \tag{3}$$

3. EIGENVALUE-BASED SPECTRUM AND ENTROPY OF DIRECTED CYCLES

3.1 Eigenvalue-Based Spectrum of Directed Cycles

The characteristic polynomials of directed cycles can be written as follows:

$$f\left(x\right)=\begin{vmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{vmatrix}$$

$$=c_0+c_1x+c_2x^2+c_3x^3+\cdots+c_{n-1}x^{n-1}.$$

The determinant of the above matrix is determined by the elements in the first row. The coefficients c_i $\left(i\in\left\{1,2,\cdots,n\right\}\right)$ are the elements of the first row of the circulant matrix (Yazlik & Taskara, 2012). It can be denoted by $J_n=Circ_n\left(c_1,c_2,c_3,\cdots,c_n\right)$, or $C_n\left(c_1,c_2,c_3,\cdots,c_n\right)$, where $J_{jk}=c_{k-j+1}$ mod

n . Specifically, $C_1\left(0,1,0,\cdots,0\right)$ is the basic circulant matrix (Cybenko & Loan, 1986). The order $n \times n$ matrix of the basic cycle is denoted by J , namely,

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4}$$

If the second vertex from the start vertex to the right vertex $\text{is}1$, then the cyclic matrix is

$$J^2 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{5}$$

Likewise,

$$J^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I. \tag{6}$$

Here I is the identity matrix. Let λ be the eigenvalue of the basic circulant matrix (Trench, 1989; Trench, 1991), then the characteristic equation of the basic circulant matrix is:

$$JX = \lambda X$$
$$\Rightarrow \left|\lambda^n I^n - J^n\right| X^n = 0$$

$$\Rightarrow \left|\lambda^n I^n - I\right| = 0\,. \tag{7}$$

Hence, the characteristic polynomial of the fundamental circulant matrixis $f\left(x\right) = \lambda^n - 1$ (Trench, 1991; Trench, 1993).

One can calculate the eigenvalues analyticaly by Fourier transformation. The eigenvalues of the digraph can be represented by the following equation:

$$\lambda_k = \sum_{k=1}^{n} e^{i\frac{2\pi}{n}k}, k = 1, \cdots, n. \tag{8}$$

λ_k is n unit eigenvalues of the characteristic equation $\lambda^n - 1 = 0$.

Theorem 1. Some properties of circulant matrix are presented as follows:

1. If both J_1 and J_2 are cyclic matrices, then $J_1 + J_2$ is a cyclic matrix;
2. If the basic cyclic matrix J is a n order matrix and J^T is its transpose cyclic matrix, then $JJ^T = J^T J = I_n$;
3. The fundamental cyclic matrix is orthogonal;
4. For the basic cyclic matrix J , $|J| = 1$ or $|J| = -1$ holds;
5. If J is a fundamental order n cyclic matrix, then there exists a natural number m such that $J^m = I$;
6. The eigenvalues of the basic cyclic matrix are complex eigenvalues of unit cycles.

Examples of circulant matrix are given below.

Example. If the network is the k-nearest neighbor coupled directed network. Let the k of the k-nearest neighbor be2 , then the adjacency matrix of the k-nearest neighbor coupled directed network is formulated as follows:

$$C_k = C_2 = \begin{pmatrix} 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 1\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix} = J + J^2. \tag{9}$$

It can be observed that the matrix in formula (9) is a circulant matrix.

So the eigenvalue of C_2 is:

$$\lambda\left(C_2\right) = \lambda_1 + \lambda_2 = e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} = e^{\frac{2\pi}{n}i} + e^{\frac{4\pi}{n}i}. \tag{10}$$

The eigenvectors of the cyclic matrix C_2 is given by

$$u_1 = \frac{1}{\sqrt{1}}(1, \omega^1, \omega^2, \cdots, \omega^{(n-1)1})^T,$$

$$u_2 = \frac{1}{\sqrt{2}}(1, \omega^2, \omega^4, \cdots, \omega^{(n-1)2})^T.$$

The nearest neighbor number is $k = 2$, and the matrix C_2 is the sum of the basic cyclic matrix $J + J^2$. So the nearest number is k in the nearest neighbor coupling network. Namely,

$$C_k = J + J^2 + \cdots + J^k. \tag{11}$$

So the eigenvalue of C_k is:

$$\lambda\left(C_k\right) = \lambda_1 + \lambda_2 + \cdots + \lambda_k = e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} + \cdots + e^{\frac{k*2\pi}{n}i} \tag{12}$$

The eigenvectors of the circulant matrix C_k are given as follows:

$$\begin{aligned} u_1 &= \frac{1}{\sqrt{1}}(1, \omega^1, \omega^2, \cdots, \omega^{(n-1)1})^T, \\ u_2 &= \frac{1}{\sqrt{2}}(1, \omega^2, \omega^4, \cdots, \omega^{(n-1)2})^T, \\ &\cdots\cdots\cdots\cdots, \\ u_k &= \frac{1}{\sqrt{k}}(1, \omega^k, \omega^{2k}, \cdots, \omega^{(n-1)2k})^T. \end{aligned}$$

Theorem 2. Let C_k be the adjacent matrix of the k-nearest neighbor coupled directed network. The characteristic polynomial is $C_k x = \lambda x$. According to equation (12), the equation $\lambda\left(C_k\right) = \lambda_1 + \lambda_2 + \cdots + \lambda_k = e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} + \cdots + e^{\frac{k*2\pi}{n}i}$ holds. We obtain the relationship between the in-degree adjacent and Laplacian matrix C_k of the k-nearest neighbor coupled directed network as follow:

$$L^- = (-1)^n C_k\left(-\lambda + k\right).$$

Hence, we obtain eigenvalues of in-degree Laplacian matrix:

$$\lambda\left(L^-, C_k\right) = (-1)^n \left(e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} + \cdots + e^{\frac{k*2\pi}{n}i}\right) + nk. \tag{13}$$

Similarly, the in-degree signless Laplacian matrix of the k-nearest neighbor coupled directed network is:

$$Q^- = C_k\left(\lambda - k\right).$$

Eigenvalues of in-degree signless Laplacian matrix is:

$$\lambda\left(Q^{-}, C_{k}\right) = \left(e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} + \cdots + e^{\frac{k*2\pi}{n}i}\right) - nk. \tag{14}$$

Proof:

According to the characteristic polynomial of the adjacency matrix of the directed cycles,

$$C_{k}x = \lambda x \Rightarrow \left|\lambda I - C_{k}\right| x = 0$$
$$\Rightarrow \left|\lambda I - \left(J + J^{2} + \cdots + J^{k}\right)\right| x = 0$$
$$\Rightarrow \left|\lambda I - J - J^{2} - \cdots - J^{k}\right| = 0\,.$$

The in-degree matrix of the k-nearest neighbor coupled directed network is denoted by D^{-}. Since $D^{-} = kI$ and $\lambda I + kI = \left(\lambda + k\right) I$, the in-degree Laplacian matrix L^{-} of k-nearest neighbor coupled directed network can be calculated as follows:

$$L^{-} = \lambda I - D^{-} + C_{k}$$
$$\Rightarrow \left|\lambda I - \left(kI\right) + \left(J + J^{2} + \cdots + J^{k}\right)\right| x = 0$$
$$\Rightarrow \left|\lambda I - kI + J + J^{2} + \cdots + J^{k}\right| x = 0$$
$$\Rightarrow \left|\left(\lambda - k\right) I + C_{k}\right| x.$$

So,

$$L^{-} = (-1)^{n} C_{k}\left(-\lambda + k\right).$$
$$\lambda\left(L^{-}, C_{k}\right) = -\lambda_{1} + k - \lambda_{2} + k - \cdots + k - \lambda_{k} + k$$
$$= (-1)^{n}\left(e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} + \cdots + e^{\frac{k*2\pi}{n}i}\right) + nk\,.$$

The equation (13) holds. Let the in-degree signless Laplacian matrix be denoted by Q^{-} for k-nearest neighbor directed network. Then,

$$Q^{-} = \lambda I - D^{-} - C_{k}$$
$$\Rightarrow \left|\lambda I - \left(kI\right) - \left(J + J^{2} + \cdots + J^{k}\right)\right| x = 0$$
$$\Rightarrow \left|\lambda I - kI - J - J^{2} - \cdots - J^{k}\right| x = 0$$
$$\Rightarrow \left|\left(\lambda - k\right) I - C_{k}\right| x.$$

So,

$$Q^{-} = C_{k}\left(\lambda - k\right).$$

$$\lambda\left(Q^{-}, C_{k}\right) = \lambda_{1} - k + \lambda_{2} - k + \cdots - k + \lambda_{k} - k$$

$$= \left(e^{\frac{2\pi}{n}i} + e^{\frac{2*2\pi}{n}i} + \cdots + e^{\frac{k*2\pi}{n}i}\right) - nk.$$

Hence, the equation (14) holds. Proof completed.

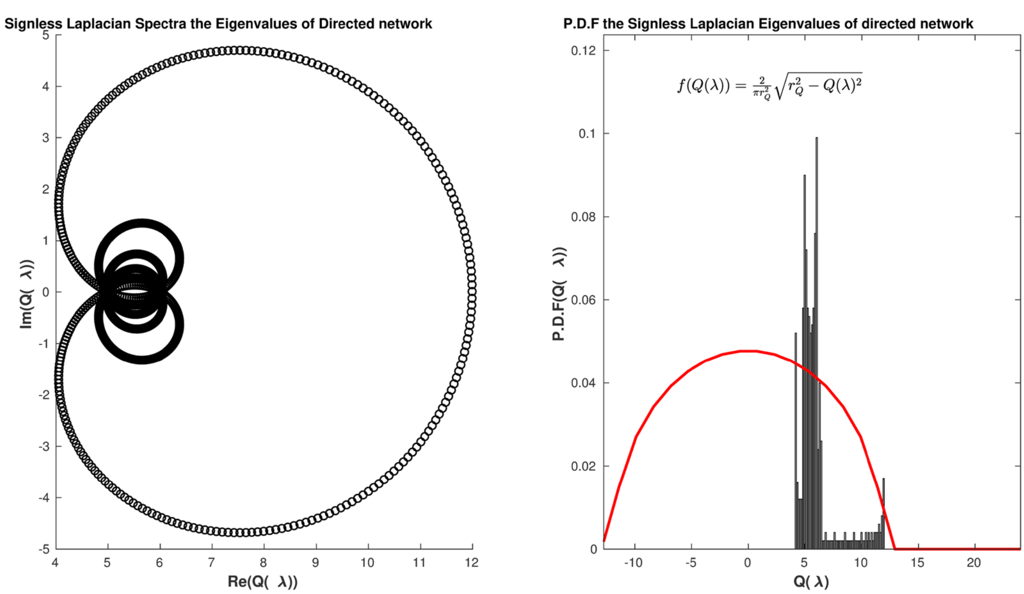
3.2 Distribution of Signless Laplacian Eigenvalue of Nearest Coupled Directed Cycles

According to Theorem 2, the eigenvalues of the three kinds matrices only differ by nk. To keep things simple, we illustrate the eigenvalue distribution using the eigenvalue of the in-degree signless Laplacian matrix. We provide a 1000-vertex directed cycles for this study. The study focuses on the parameter k of the nearest neighbor coupled network in directed cycles. On one hand, k represents the nearest neighbor, as shown in Figure 1. On the other hand, k also represents vertex intervals.

For example, $k = 6$ corresponds to the sixth neighbor in the network. In Figure 1, each node is connected to six neighboring nodes in the network, resulting in six cycles.

The nearest neighbor coupled directed network model is a cyclic graph with n vertices, and its adjacency matrix exhibits translation symmetry (Yazlik & Taskara, 2012) in the constructed network (Aliaksei & Moura, 2013). In this model, each vertex is connected to at most k of its neighbors, and the edges are directed in one direction.

Figure 1. Spectrum of in-degree signless Laplacian of a directed nearest neighbor coupled network (k=6, n=1000)



On the other side of this work, we investigate the properties of the eigenvalue-based in-degree signless Laplacian for different vertex intervals, specifically $k = 2$, 3, 4, 5, 6, and 10. For the case of $k = 2$, the eigenvalue of the nearest neighbor coupled network $C_{k=2}$ is calculated as follows:

$$\lambda\left(Q^{-}, C_{k}\right) = \lambda_{1} + \lambda_{3} + \lambda_{5} + \cdots + \lambda_{m*k+1} = \left(e^{\frac{2\pi}{n}i} + e^{\frac{3*2\pi}{n}i} + \cdots + e^{\frac{\left(k*m+1\right)*2\pi}{n}i}\right) - nk, \tag{15}$$

$\left(m = 0, 1, \cdots, n-2\right)$. Then the spectrum of the nearest neighbor coupling network is shown in Figure 2.

When the intervals $k = 3$, the eigenvalue of the nearest neighbor coupled network $C_{k=3}$ is,

$$\lambda\left(Q^{-}, C_{k}\right) = \lambda_{1} + \lambda_{4} + \lambda_{8} + \cdots + \lambda_{m*k+1} = \left(e^{\frac{2\pi}{n}i} + e^{\frac{4*2\pi}{n}i} + \cdots + e^{\frac{\left(m*k+1\right)*2\pi}{n}i}\right) - nk, \tag{16}$$

$\left(m = 0, 1, \cdots, n-3\right)$. When intervals $k = 3$, the spectrum of the nearest neighbor coupling network is shown in Figure 3.

In the case $k = 4$, the eigenvalue of the nearest neighbor coupled network $C_{k=4}$ is,

$$\lambda\left(Q^{-}, C_{k}\right) = \lambda_{1} + \lambda_{5} + \lambda_{9} + \cdots + \lambda_{m*k+1} = \left(e^{\frac{2\pi}{n}i} + e^{\frac{5*2\pi}{n}i} + \cdots + e^{\frac{\left(k*m+1\right)*2\pi}{n}i}\right) - nk, \tag{17}$$

Figure 2. Spectrum of in-degree signless Laplacian of a directed nearest neighbor coupled network (k=2, n=1000)

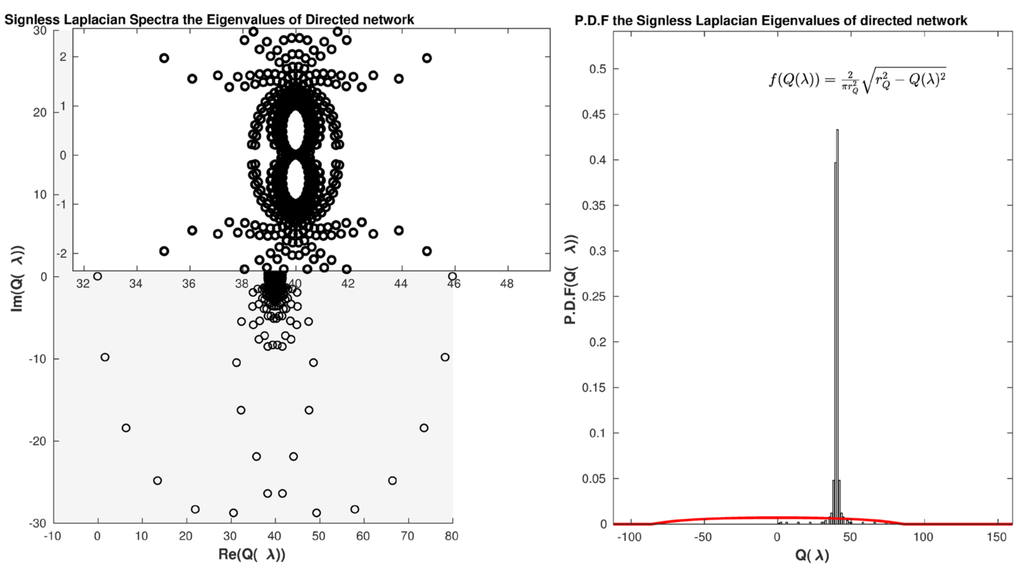
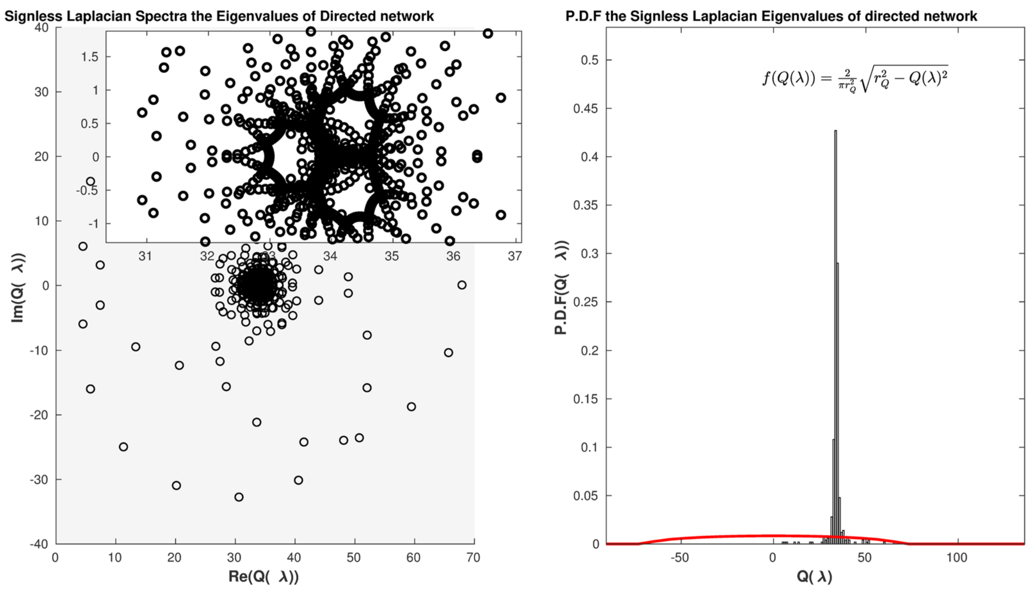


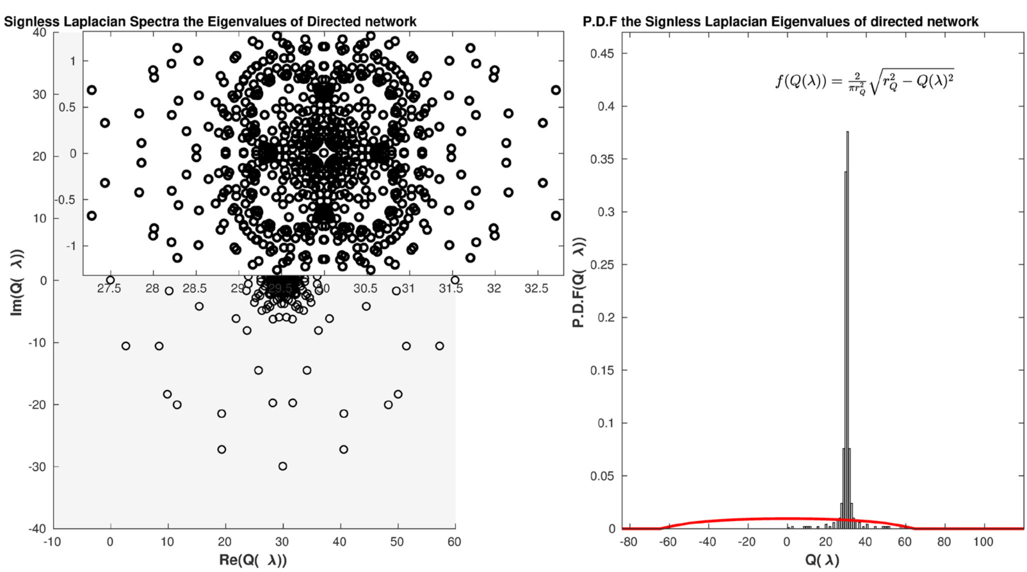
Figure 3. Spectrum of in-degree signless Laplacian of a directed nearest neighbor coupled network (k=3, n=1000)



$\left(m = 0,1,\cdots,n-4\right)$. Then the spectrum of the nearest neighbor coupling network is shown in Figure 4.

If the interval of the cyclic directed network vertex is $k = 5$, which means skipping 5 vertex to select the neighbor vertex, namely taking $\left\{1,6,11,\cdots,m\times 5+1\right\}$, then spectrum of the nearest

Figure 4. Spectrum of in-degree signless Laplacian of a directed nearest neighbor coupled network (k=4, n=1000)



neighbor coupled network is shown in Figure 5. In the case $k=5$, the eigenvalue of the nearest neighbor coupled network $C_{k=5}$ is,

$$\lambda\left(Q^{-}, C_{k}\right)=\lambda_{1}+\lambda_{6}+\lambda_{11}+\cdots+\lambda_{m*k+1}=\left(e^{\frac{2\pi}{n}i}+e^{\frac{6*2\pi}{n}i}+\cdots+e^{\frac{(k*m+1)*2\pi}{n}i}\right)-nk, \quad (18)$$

$\left(m=0,1,\cdots,n-5\right)$.

If the interval is $k=10$ in the cyclic directed network, which means skipping 10 vertex to select the neighbor vertex, namely taking $\left\{1,10,20,\cdots,m\times 10+1\right\}\left(m=1,10,20,\cdots,n-10\right)$, then the spectrum of the nearest neighbor coupled network is shown in Figure 6.

In Fig.1 $\sim$ Fig.6, on the left of figures is the relationship between the real part and the imaginary part of the eigenvalues of the nearest neighbor coupled network. On the right of figures is the Probability Density Function (PDF) of a eigenvalue, and red lines show Wigner's Semicycles Law (Farkas et al., 2001) $\left(p=O.5, \sigma^{2}=0.25\right)$.

3.3 Eigenvalue-Based Entropy of Directed Cycles

Due to the asymmetry of the directed network matrix, most of its eigenvalues are complex numbers, including both positive and negative values. In this study, we propose a novel entropy measure based on the in-degree eigenvalues of the adjacency matrix, in-degree Laplacian matrix, and in-degree signless Laplacian matrix. Let Re denote the real part entropy. Since i represents the imaginary unit, we use $\left|\lambda_{j}\right|$ to represent the absolute value of the j-th eigenvalue of the adjacency matrix. Similarly, $\left|\mu_{j}\right|$ represents the absolute value of the j-th eigenvalue of the in-degree Laplacian matrix,

Figure 5. Spectrum of in-degree signless Laplacian of a directed nearest neighbor coupled network (k=5, n=1000)

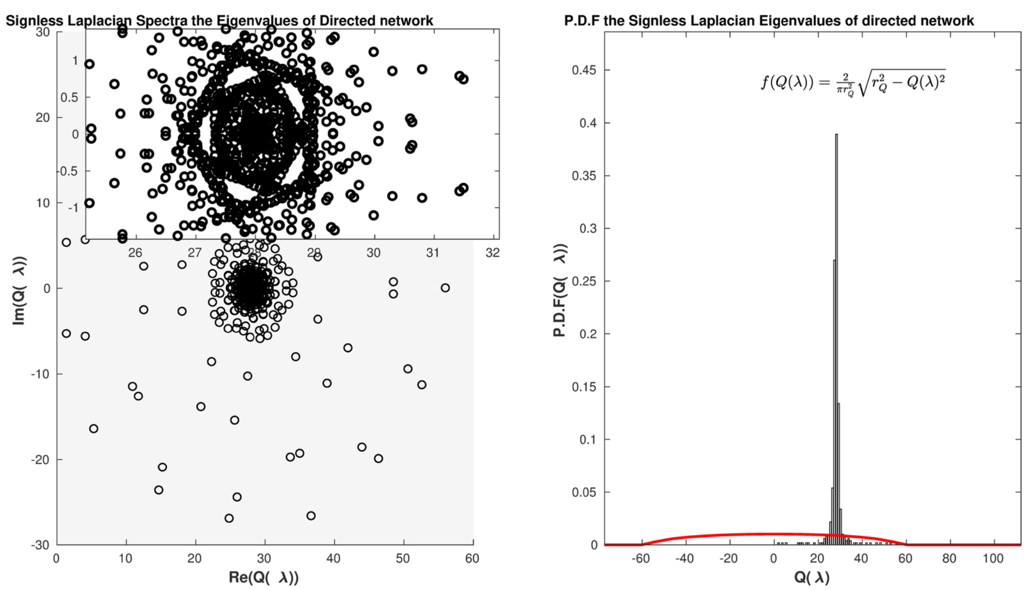
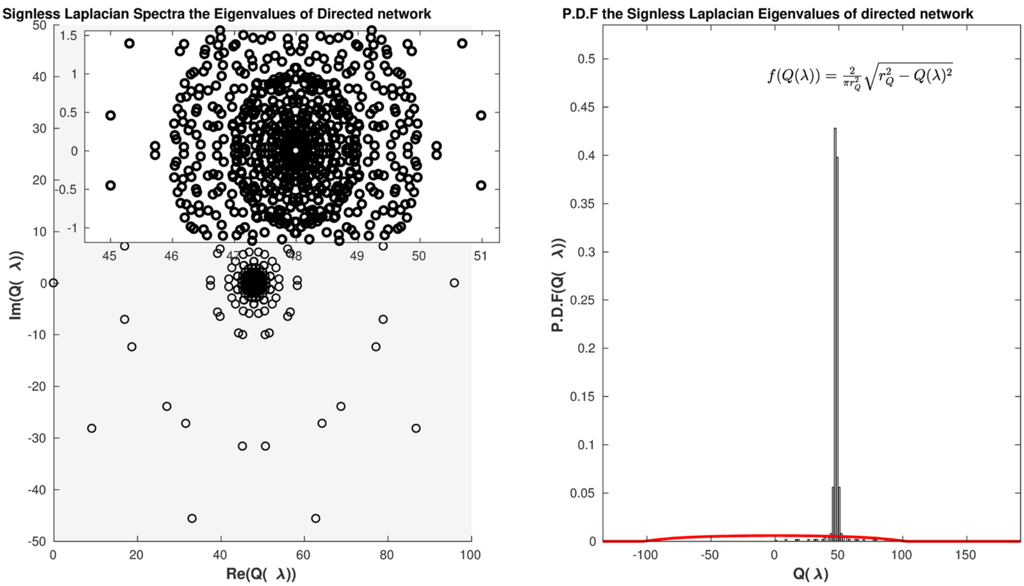


Figure 6. Spectrum of in-degree signless Laplacian of a directed nearest neighbor coupled network (k=10, n=1000)



and $\left|q_j\right|$ represents the absolute value of the j-th eigenvalue of the in-degree signless Laplacian matrix for digraph.

Next, we define the real part eigenvalue-based entropy for these three kinds matrices of directed networks in the following.

Definition 5. The eigenvalue-based real part entropy of adjacency matrix for digraph is defined as:

$$I\left(Re\left(A^{-}\right)\right) = -\sum_{j=1}^{n} \frac{\left|Re\left(\lambda_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(\lambda_k\right)\right|} log \frac{\left|Re\left(\lambda_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(\lambda_k\right)\right|}.$$

Definition 6. The eigenvalue-based real part entropy of in-degree Laplacian matrix for digraph is defined as:

$$I\left(Re\left(L^{-}\right)\right) = -\sum_{j=1}^{n} \frac{\left|Re\left(\mu_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(\mu_k\right)\right|} log \frac{\left|Re\left(\mu_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(\mu_k\right)\right|}.$$

Definition 7. The eigenvalue-based real part entropy of in-degree signless Laplacian matrix for digraph is defined as:

$$I\left(Re\left(Q^{-}\right)\right) = -\sum_{j=1}^{n} \frac{\left|Re\left(q_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(q_k\right)\right|} log \frac{\left|Re\left(q_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(q_k\right)\right|}.$$

We obtained the eigenvalue-based real part entropy of three kinds matrices of the directed cycles. Let $k = 1$, the eigenvalues are transformed by the complex Euler formula,

$$\lambda_1 = e^{i\frac{2\pi}{n}1} = e^{i\frac{2\pi}{n}} = cos\left(\frac{2\pi}{n}\right) + isin\left(\frac{2\pi}{n}\right).$$

Extending to the general case,

$$\lambda_k = \sum_{k=1}^{n} e^{i\frac{2\pi}{n}k} = \sum_{k=1}^{n} cos\left(\frac{2\pi}{n}k\right) + \sum_{k=1}^{n} isin\left(\frac{2\pi}{n}k\right).$$

$$I\left(Re\left(A^-\right)\right) = -\sum_{j=1}^{n} \frac{\left|Re\left(\lambda_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(\lambda_k\right)\right|} log \frac{\left|Re\left(\lambda_j\right)\right|}{\sum_{k=1}^{n}\left|Re\left(\lambda_k\right)\right|} = \sum_{j=1}^{n} \frac{\left|cos\left(\frac{2\pi}{n}j\right)\right|}{\sum_{k=1}^{n}\left|cos\left(\frac{2\pi}{n}k\right)\right|} log \frac{\left|cos\left(\frac{2\pi}{n}j\right)\right|}{\sum_{k=1}^{n}\left|cos\left(\frac{2\pi}{n}k\right)\right|}.$$

According to equations (13) and (14), we obtain,

$$I\left(Re\left(L^-\right)\right) = nk + (-1)^n \sum_{j=1}^{n} \frac{\left|cos\left(\frac{2\pi}{n}j\right)\right|}{\sum_{k=1}^{n}\left|cos\left(\frac{2\pi}{n}k\right)\right|} log \frac{\left|cos\left(\frac{2\pi}{n}j\right)\right|}{\sum_{k=1}^{n}\left|cos\left(\frac{2\pi}{n}k\right)\right|}.$$

$$I\left(Re\left(Q^-\right)\right) = \sum_{j=1}^{n} \frac{\left|cos\left(\frac{2\pi}{n}j\right)\right|}{\sum_{k=1}^{n}\left|cos\left(\frac{2\pi}{n}k\right)\right|} log \frac{\left|cos\left(\frac{2\pi}{n}j\right)\right|}{\sum_{k=1}^{n}\left|cos\left(\frac{2\pi}{n}k\right)\right|} + nk.$$

4. EXPERIMENTAL SIMULATION OF EIGENVALUE-BASED ENTROPY

Table 1 gives the eigenvalue-based entropy values of the real part in the nearest neighbor coupled network for $k = 2, 3, 4, 5, 6$.

Table 2 shows the eigenvalue-based real parts entropy values of the three matrix when the nearest neighbor number are invervals $k = 10, 125, 175, 225, 350, 500$.

Table 1 and Table 2 show eigenvalue-based real part entropy in three kinds matrices. The evolution law of directed cycles is verified in Table 1 and Table 2.

5. CONCLUSION

In this work, we study spectrum and the eigenvalue-based real part entropy of complex in directed cycles complex digraph. We visualize the spectral spectrum structure of in-degree signless Laplacian matrices.

Table 1. Eigenvalue-based real parts entropy values of the three kinds matrix in the nearest neighbor coupled network ($k = 2,3,4,5,6$)

Nearest Neighbor Coupling k	2	3	4	5	6
$I\left(Re\left(A^{-}\right)\right)$	5.5969	5.7588	5.8024	5.8286	5.3525
$I\left(Re\left(L^{-}\right)\right)$	6.9002	6.8993	6.8983	6.8976	6.9038
$I\left(Re\left(Q^{-}\right)\right)$	6.9002	6.8996	6.8983	6.8977	6.9038

Table 2. Eigenvalue-based entropy values of the real parts of three kinds matrix in the nearest neighbor coupled network $\left(k = 10,125,175,350,500\right)$

Nearest Neighbor Coupling k	10	125	175	350	500
$I\left(Re\left(A^{-}\right)\right)$	5.6385	6.1739	6.5195	6.6091	6.7630
$I\left(Re\left(L^{-}\right)\right)$	6.9018	6.8364	6.8130	6.7644	6.6009
$I\left(Re\left(Q^{-}\right)\right)$	6.9018	6.8364	6.8130	6.7644	6.6009

We obtain new interpretation of the nearest neighbor vertex k , which denotes the interval of the directed cycles network. Namely, k is equivalent to hopping k step to the neighbor vertex, m is the total number of vertices of the network divided by k .

Simulation experiments demonstrate that in-degree signless Laplacian spectrum and real numerical solution the eigenvalue-based real part entropy of directed cycles in complex digraph.

The work is helpful to deepen the understanding of GCN models (Aliaksei & Moura, 2013; Newman, 2001). It can also be applied to social computing. A regular model framework can be established for exploring the aggregation properties of distributions for classification of games.

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