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The Robustness and Vulnerability of a Complex Adaptive System With Co-Evolving Agent Behavior and Local Structure

Xiaojing Zheng, Weifang University of Science and Technology, China*

ABSTRACT

Agents' irrational behavior would lead to local configuration of complex adaptive system percolation. The corresponding critical point is key to making decisions for improving the system or keeping the system from collapsing. The authors construct a complex adaptive system model where agent behavior and its local configuration co-evolve. This model shows, when an arbitrary agent and its neighbors change their strategy and local interactive configuration, how the properties of percolation critical point of this system would emerge under random attack and intentional attack. It is shown that the system is robust if it is attacked randomly, and there are always at least two large components keeping the system connected. However, if the system is attacked intentionally, the result is more interesting. The system is robust if the deleting probability is smaller than a certain critical probability c_0, but the system is vulnerable if the probability is larger than c_0. Furthermore, the critical probability c_0 is determined by the agent payoff, system structure, and noise.

KEYWORDS

co-evolutionary complex adaptive system, percolation, robustness, vulnerability

1. INTRODUCTION

Although individual behavior is too complex to fully explain, collective behavior satisfies certain laws. According to Thomas C. Schelling, the 2005 Nobel Prize Winner in Economics, "sometimes the results are surprising, sometimes they are not easily guessed, sometimes the analysis is difficult" (Schelling, 2006). This point was emphasized by scientist Philip Ball in his book entitled *Critical Mass: How One Thing Leads to Another*, "collective behaviors are not necessary equal to the linear sum of behaviors of all persons, …, it can be transferred sharply to the totally inverse aspect even if few individuals change their behaviors slightly" (Ball, 2004) . This phase transition phenomenon is called percolation. Percolation makes the system flourish or collapse suddenly if it is at the intermediate state between order and disorder (Bakir, Tarrasy, et.al., 2016). For example, it is difficult to transition

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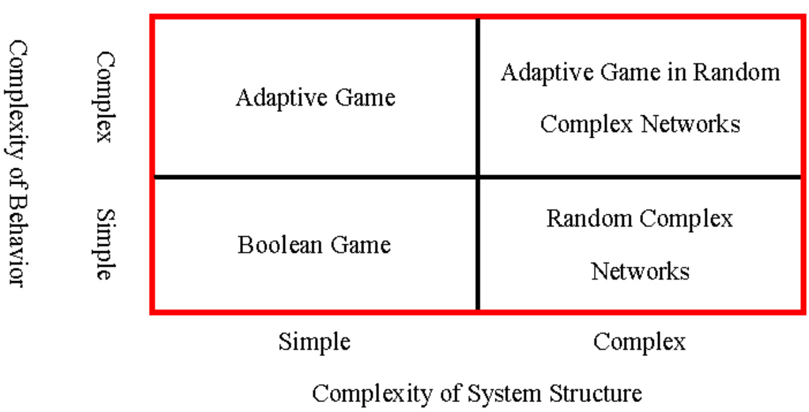
*Corresponding Author

from few accepting a new technology to its widespread public acceptance (Friedkin et.al., 2016; Chen, J. et.al., 2021; Chen, S. et.al., 2021). The technology is suddenly accepted by the wider public if, and only if, the corresponding acceptance probability reaches a certain threshold (Wu, L.F. et.al., 2019). The collapse of China's stock market in 2015 is another example. A-shares suffered a crash during the 52 trading days between June 15 and August 26. Only 102 of the total A-shares rose and as many as 2,498 dropped, of which 1,541 dropped by more than 50 percent (the data coming from https://xueqiu.com/4292490144/50196986). Considering these examples and circumstances, we would note that there seems to be a special attractor that attracts the system running toward this "percolation" point. It is also seen that percolation exists pervasively, such that the system always evolves around the critical point. This seems to be an attractor that attracts the system to move toward a certain criticality. In this paper, we ask: To achieve system percolation, how many individuals need to change their behaviors, and who are the individuals whose behaviors need to change?

According to scholars, the complexity of a system's criticality of percolation relies on the properties of the system. In turn, the system's properties are deterministic under certain interactive rules between agents and between agents and their environment. In this sense, four categories of system, according to agent behavior and system structure, are defined in Figure 1.

Figure 1 gives the impression that percolation is the exclusive phenomenon of a complex system whose structure is large-scale. Because of analyzing complexity, scholars simplify the interaction between individuals to a Boolean game, where the relationship with other is denoted as 1 if they frequently play together or 0 if any other outcome. In this sense, a broad range of percolation phenomena have been studied from abstract models (Misra et al., 2010; Bollobás et al., 2007, 2012; Berche et al., 2004; Fumiya et al., 2011; Solé et al., 2015) and from physical systems (Ostrom, 2009; Barkoczi & Galesic, 2015; Galesic et al., 2015). In the literature, two common kinds of attacks have been considered, the random attack and the intentional attack. In the random attack, $(1-p)n$ agents are removed from the system randomly. In the intentional attack, the cn strongest agents are removed from the system. The complex adaptive system (CAS) would be destroyed sharply if another p' less than p or c' larger than c are taken away. The corresponding parameters of p and c are all called the critical probability to random attack and intentional attack, respectively (Afra, 2004; Brauer et al., 2010; Baxter et. al., 2011; Fumiya & Kousuke, 2011; Callway et al., 2000; Albert et. al., 2000). Similar conclusions have been obtained by Derzhko, (2004); Berche et al., (2004); Nogueira et al., (2008); and Barenblatt et al. (1996). Their results show that inhomogeneous random complex networks are robust if attacked randomly; however, there exists a critical probability c when attacked intentionally. If the probability of attack is slightly larger than c, the system is vulnerable. If the

Figure 1. Interactivity between agents in a system



probability is slightly smaller than c, the system is robust, and if the probability is equal to c, the system is critical. In practice, however, we must account not only for the individual behavior in the system, but also for the system structure that is random and time-varying (Alessandretti, et.al., 2020; Ehlert, et.al., 2020; Dezfouli, et.al., 2020; Vedadi & Greer, 2021). These all share the characteristic that it is difficult to explain this CAS exactly from a mathematical viewpoint. However, the interaction between agents is more complex than what has been discussed above, which makes the property of critical probability c more complex. In fact, the CAS would be defined as follows:

Hypothesis 1. *There are several local-worlds (a connected sub-graph) in CASs. They are small enough, from the macroscopic space-scale, such that there are more than enough local-worlds to interact together in a system, and they are large enough, from the microscopic space-scale, such that there are more than enough agents to interact together in a local-world.*

Hypothesis 2. *The interactive behavior between agents in a certain local-world is defined as being the cooperative adaptive game; however, the interactive behavior between agents in different local-worlds is defined as the non-cooperative game.*

Hypothesis 3. *There are several short time-scales in the system evolution process. The system structure is stable in each short time-scale and is dynamic because the individuals in the system adaptively adjust to its local topological configuration to obtain more benefits.*

This complexity, coupled with time-varying behavior and system configuration, makes the percolation criticality of a system more difficult. Note that the thresholds of dividing macroscopic/microscopic space-scale and short/long space-scale can be seen from Ehlert, Kindschi, et.al., (2020) and Nogueira and Kleinert, (2004). To make the analysis feasible, we assume that the system changes more slowly than agents' behavior (Tump et.al., 2020; Baldassarri &Abascal, 2020; Chen &Rohla, 2018). Then, we regard the behavior dynamics equation as a variable because each agent makes decisions by relying on local information matched to the agent's local configuration. This can reduce the analysis complexity, as described in the following section. Furthermore, the complexity comes from the agent's behavior of selecting partners. In this paper, each agent selects a fitness player to interact with according to a preferential attachment with payoff. Additionally, the payoff, which can be defined as both income, within market paradigm, that can be measured by money and benefits, and within social paradigm, that cannot be measured by money, is determined by agent strategy α and local topological structure ω, where ω is the local information of interaction graph between agents. But in classic research, topological structure is described just as degree or strength of the graph. No matter who is referred to, the degree or strength is explicitly connected with graph structure. Unfortunately, the payoff is explained implicitly, which is beyond our current knowledge. To make it feasible, one must notice that the preferential attachment of payoff is equivalent to the one of strength decided by weight. The system structure is fixed if the time-scale is smaller than a certain threshold, defined as $t \leq t_c$ in the next section. Then, the agent's payoff could converge to the corresponding attractor $\tilde{\pi}$, where the evolution time holds an exponential distribution $y = \exp(-at)$, where a is a positive constant. In fact, each agent's payoff attractor is similar to the corresponding strength. Given our hypotheses, the system could be resolved by using the analysis methods from weighted random complex networks.

In this paper, a theoretical model of CASs is constructed to describe the system's properties. Then, the percolation criticality of the theoretical random attack and intentional attack models is constructed by introducing the percolation theory and its corresponding method. The remainder of this paper's scope is as follows. In section two, we construct a generalized model for a CAS to describe the system's property, nature, state, and evolution. However, this model is not solvable; therefore, an equivalence model, with a continuous stochastic process, is specified in section four. The proof of identity with the initial model is then specified and resolved in this section. Next, we demonstrate that the criticality of growth is equivalent to the one of decay. For convenient analysis, we focus on the continuous model. Then, the random attack and intentional attack to the system are studied, followed by a calculation of the analytic critical state ad of the conditions of robustness and vulnerability. We then present the corresponding mathematical percolation criticality.

2. THE CO-EVOLUTIONARY CAS

Individual behavior in society is diverse and random, and so is the interactive relationship between agents. To describe its properties, one must assume that people can interact with others under a series of limited conditions. As specified in Hypotheses 1-3, this structure for the CAS satisfies hierarchical structure; furthermore, there are several local-worlds in each layer, which reflects the complex structure of CASs. Individuals in the same local-world with similar properties are defined as homogenous agents. All others are called inhomogenous agents. Similarly, the interactions between agents are defined as the cooperative game and the non-cooperative game, respectively (see Figure 2).

As shown in Figure 2, C represents the cooperative adaptive game and N represents the non-cooperative adaptive game. More importantly, social economical organization is full of dissipative structure, which makes the structure of the system changeable (Lee, Karimi, et.al., 2019), as shown in Figure 3.

At time t, interactive configuration is presented on the left side of Figure 3. However, at time $t+1$, it is changed to the right side of Figure 3. There is a totally different structure, where agents 1, 2, 3, 4, and 7 are in subsystem 1 and agents 5, 6, 8, and 9 are in subsystem 2.

If an individual makes a decision, strategies and local configuration must be acknowledged. The former consists of his current and historic strategies, neighbors' strategies, and the property of the

Figure 2. Interaction between agents

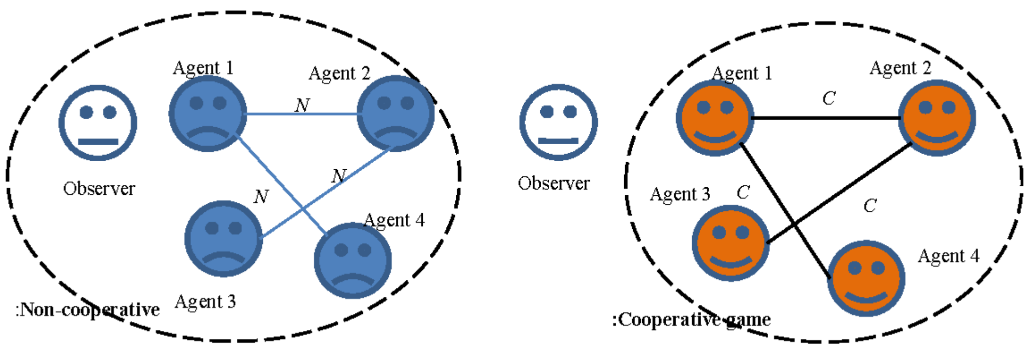
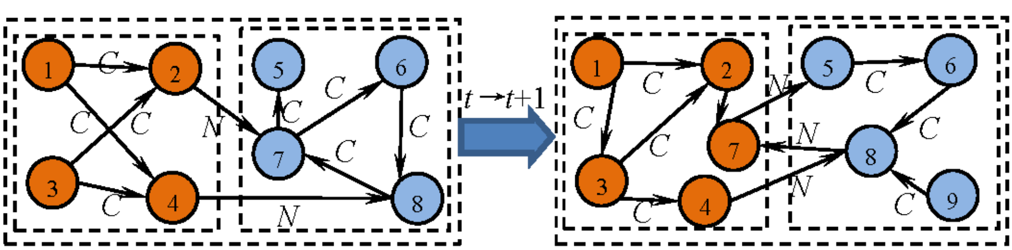


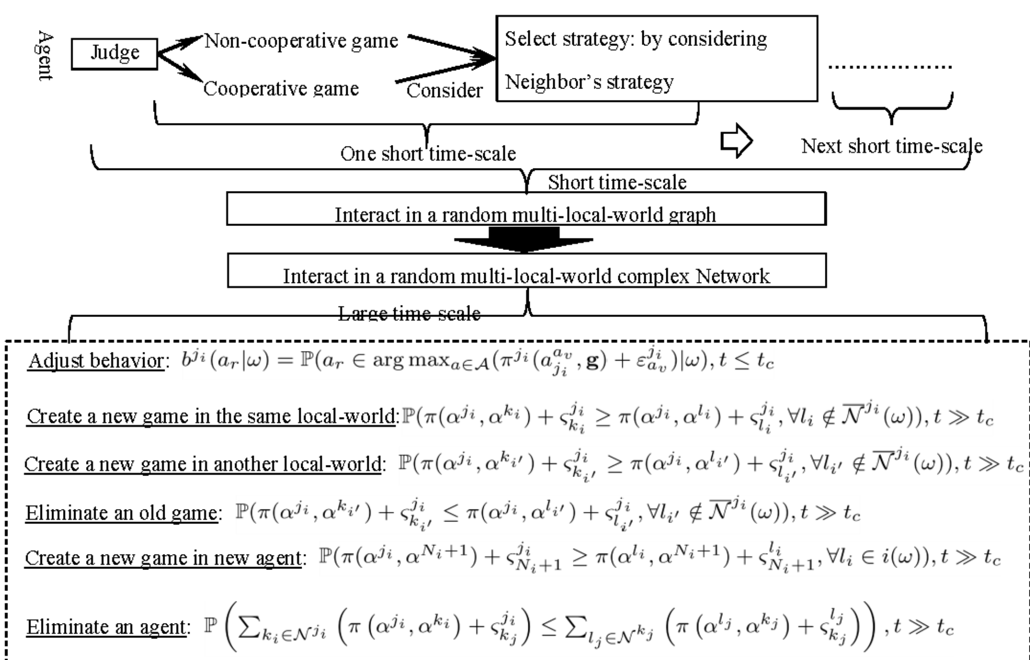
Figure 3. Interaction between agents in the dynamic topology of the CAS



environment, and the latter means his local topological structure. The local topological configuration is stable if and only if each the conditions of perfect information is satisfied (Zheng, et.al., 2012; Zheng, et.al., 2016). In this sense, the optimal strategy of each agent would be to converge into a certain interval (See Zheng, et.al., 2012, the Appendix A of this article). On the other side, the optimal strategies would be evolving with its local topological configuration, because they would select more suitable partners adaptively when time-scale is shifted from one time-scale to the next. Each of the following behaviors could happen: (1) create a new game with a stranger in the same local-world, in other local-worlds, or with a new agent who joined in the system; (2) eliminate an existing game from an existing agent; and (3) quit the game (See the Appendix B of this article). However, these optimal time-varied strategies would satisfy a Poisson distribution, as discussed by Zheng, et.al., (2012) and as described in the Hypotheses 1-3. Furthermore, no matter what event happened, an arbitrary agent would change its local topological configuration, which results in a system's structure changing dynamically. Set j_i represents the j^{th} agent of local-world i, $\pi(.)$ is its payoff converged in the corresponding short time-scale, α is its strategy, $\mathbf{g}$ is the local topological configuration, $\mathcal{N}$ is the set of all neighbors of agent j_i, $\overline{\mathcal{N}}$ is the union set of $\mathcal{N}$ and agent j_i. This process in the long time-scale is as described in Figure 4.

A theoretical model for a CAS is constructed to describe the system's properties. Suppose that there are l subsystems in the system, and the threshold for the average short time-scale is t_c (there exists a constant $\lambda \sup_{\alpha\in\mathcal{A}} \pi(t) \le E[\pi(t)+\lambda]$ and a time threshold t_c such that for any $t \ge t_c$, hold). In an arbitrary short time-scale, arbitrary agent j_i would randomly select one of six behaviors if its game radius r is driven by reachable agents and length, L is driven by the historic game knowledge. It can adjust its behavior, create a new game relationship in the same sub-system, create a new game

Figure 4. Agent's behavior over a long time-scale



relationship into another sub-system, eliminate an old game relationship, create a game relationship with a new agent of the system, or exit the system, with probability $q_1, q_2, \ldots, q_6$ according to

$$b^{j_i}(a_r \mid \omega) = \mathbb{P}\Big(a_r \in \arg\max_{a_v \in \mathcal{A}}\Big(\pi^{j_i}\big(\alpha_{j_i}^{a_v}, \mathbf{g}\big) + \varepsilon_{a_v}^{j_i}\Big)\Big|\omega\Big), t \le t_c$$

$$\mathbb{P}\Big(\pi\big(\alpha^{j_i}, \alpha^{k_i}\big) + \varsigma_{k_i}^{j_i} \ge \pi\big(\alpha^{j_i}, \alpha^{l_i}\big) + \varsigma_{l_i}^{j_i}, \forall l_i \notin \overline{\mathcal{N}}^{j_i}(\omega)\Big), t \gg t_c$$

$$\mathbb{P}\Big(\pi\big(\alpha^{j_i}, \alpha^{k_{i'}}\big) + \varsigma_{k_{i'}}^{j_i} \ge \pi\big(\alpha^{j_i}, \alpha^{l_{i'}}\big) + \varsigma_{l_{i'}}^{j_i}, \forall l_{i'} \notin \overline{\mathcal{N}}^{j_i}(\omega)\Big), t \gg t_c$$

$$\mathbb{P}\Big(\pi\big(\alpha^{j_i}, \alpha^{k_{i'}}\big) + \varsigma_{k_{i'}}^{j_i} \le \pi\big(\alpha^{j_i}, \alpha^{l_{i'}}\big) + \varsigma_{l_{i'}}^{j_i}, \forall l_{i'} \notin \overline{\mathcal{N}}^{j_i}(\omega)\Big), t \gg t_c$$

$$\mathbb{P}\Big(\pi\big(\alpha^{j_i}, \alpha^{N_i+1}\big) + \varsigma_{N_i+1}^{j_i} \ge \pi\big(\alpha^{l_i}, \alpha^{N_i+1}\big) + \varsigma_{N_i+1}^{l_i}, \forall l_i \in i(\omega)\Big), t \gg t_c$$

$$\mathbb{P}\left(\sum_{k_i \in \mathcal{N}^{j_i}} \left(\pi\left(\alpha^{j_i}, \alpha^{k_i}\right) + \varsigma_{k_j}^{j_i}\right) \le \sum_{l_j \in \mathcal{N}^{k_j}} \left(\pi\left(\alpha^{l_j}, \alpha^{k_j}\right) + \varsigma_{k_j}^{l_j}\right)\right), t \gg t_c$$

respectively, where $1 \le i \le l, a, \alpha$ is the corresponding strategy, ς is the noise of the payoff, $\mathcal{N}^{j_i}$ is the set of agent j_i and its neighbors, $\overline{\mathcal{N}^{j_i}} \equiv \mathcal{N}^{j_i} \setminus j_i$, π is corresponding payoff (Zheng, X., & Zheng, J., 2016).

This universal model can be simplified to corresponding classic models. Set $t_c = \infty, \omega$ is constant and $l = 1$, it is an adaptive game problem. Set $t_c = 0$, $l = 1$, and if the interaction between agents is defined as a Boolean game, this model can be degenerated to the BA model (Albert, et.al, 2000). Set $l = 1$, if the agents are inhomogeneous, and if the interaction between agents is a Boolean game, this model is the LCD model (Bollobás et al., 2007; 2012; Letellier, C., 2021). Set $l < \infty$, if the Boolean game dominates, and set preferential attachment mechanism based on the in-degree and out-degree of the agents is the selected, the model would be degenerated to the multi-local-world complex networks model constructed by Li and Chen, (2003). Set $l = 1$, set the prisoner's dilemma drives the game, and set the preferential attachment mechanism is designed due to payoff of agent; this model has been previously studied (McAvoy, et.al., 2020).

Complex networks are a reduced interaction of the stochastic game, which is more complex than the Boolean game. We already know the criticality of a thriving system is equivalent to that of a collapsing system (Zheng, et.al., 2016). In this paper, the criticality is considered by random attack and intentional attack, respectively. In the random attack, $(1-p)n$ agents are removed from the system randomly. In the intentional attack, the cn strongest agents are removed from the system. A very broad range of percolation has been studied using abstract models (Misra et al., 2010; Bollobás et al., 2007; 2012; Granovette,. 1978; Dionne, et.al., 2019; Qin, et.al., 2016) and physical systems (Ostrom, 2009; Bai, et.al., 2020; Lucas & Nordgren, 2020; Centola, et.al., 2018). Common conclusion shows that there exists certain probabilities p and c corresponding to random attack and intentional attack respectively, such that the CAS would be destroyed sharply if another p' is less than p or another c' is larger than c; p and c are all called corresponding critical probability (Afra, 2004; Brauer, et al., 2010; Baxter, et al., 2011; 2009; Fumiya & Kousuke, 2011; Callway, et al., 2000; Albert, et al., 2000). According to studies of Derzhko, (2004); Berche, et al., (2004); Nogueira, et al., (2008), and Barenblatt, et al., (1996), the inhomogeneous random complex network is robust if it is attacked randomly. However, there exists a critical probability c if it is attacked intentionally. If the probability of attack is larger than c, the system is vulnerable; if the probability is smaller than

c, the system is robust. If the probability is equal to c, the system is critical. Further, Christenensen, et al., (2005) has stated that the percolation parameters only rely on a probability $\mathcal{R}_b\{p\}$.

However, they cannot fully explain agents' complex behavior from a mathematical viewpoint because their behaviors are more complex than what has been studied previously. As demonstrated by Zheng, et.al., (2012), the payoff of arbitrary agents in the system could converge to certain attractors; we will omit the details of interaction in short time-scale and just focus on the invariable distribution of the dynamic optimal strategies. In the following sections, we focus on what would happen to the CAS when an arbitrary agent is moved out for a random attack and for an intentional attack.

MAIN RESULTS AND CONCLUSION

It is concluded that the CAS is robust if it is attacked randomly. Furthermore, if arbitrary agents in this system are kept equiprobably with a very small positive number p and the other ones are deleted with probability $1-p$, the system cannot be destroyed. In this case, there are two large components that keep the system connected, set the rank of the largest component $L_1(p)$, and set the rank of the second largest component $L_2(p)$.

Theorem 3.1. There exists a function $\lambda(p) > 0$, for arbitrary $0 < p < 1$, such that $L_1(G_p) = (\lambda(p) + o(1))n$ and $L_2(G_p) = n$ hold. For a very small $p > 0$, there exists at least a giant-component in this system that makes it robust, as $p \to 0$, the threshold value of rank $\lambda(p)$ of this largest component would be

$$\exp(-\Theta(1-p^2)) \le \lambda(p) \le (1 + 5d \sup_{\alpha\in\mathcal{A}} E[\pi]p / 8) \exp\left(-1 / (\sup_{\alpha\in\mathcal{A}} E[\pi]p)\right)$$

where, $E[\pi]$ is the average payoff of transitory at an arbitrary short time-scale. $g(x) = \Theta(f(x))$ means that $g(x) = O(f(x))$ and that $f(x) = O(g(x))$. The upper bound and lower bound of the rank are important for estimating the size of the largest component when the system is destroyed randomly.

It is known from Theorem 3.1, that when the system is attacked randomly, even if there are a few agents that stay within this system, there exist two giant components that make the system connected. The maximum value of the rank of the largest component is $\left(1 + 5d \sup_{\alpha\in\mathcal{A}} E[\pi]p / 8\right) \exp\left(-1 / 2 \sup_{\alpha\in\mathcal{A}} E[\pi]p\right) n$ and the minimum value is $n \exp(-\Theta(1-p^2))$. The rank of the second largest component's size is $o(n)$.

However, this system is vulnerable when it is attacked intentionally. Even when there are a few strong agents removed with a small probability c, the system could be destroyed. The system is in criticality, as described by Theorem 3.2.

Theorem 3.2. There exists a positive constant $0 < c_0 = \dfrac{q_3 - q_4}{1 + \delta_{in}\left(q_1 + q_2 - q_5\right)} \dfrac{\inf_{\alpha\in\mathcal{A}} E[\pi] - 1}{\sup_{\alpha\in\mathcal{A}} E[\pi] + 1} < 1$,

where $\pi \ge 0$, and a positive constant $\theta(c)$, such that $L_1(G_c) = o(n)$ with probability $1 - o(1)$ if $c \ge c_0$, and $L_1(G_c) = (\theta(c) + o(1))n$ and $L_2(G_c) = o(n)$ with probability $1 - o(1)$ if $c < c_0$.

It is known from Theorem 3.2, that there exists a critical probability c such that CAS percolation happens. Furthermore, when the probability of removing agents is smaller than c, there exist two components keeping the system connected. The rank of the large one is $(\theta(c)+o(1))n$ and the rank of the small one is $o(n)$, which makes this system robust. When the probability of removing agents is larger than c, there only exists one large component and its rank is $o(n)$, which cannot make the system operate normally. When the probability is equal to c, the system has the property of criticality. Furthermore, the critical probability $c=\frac{q_3-q_4}{1+\delta_{in}\left(q_1+q_2-q_5\right)}\frac{\inf\limits_{\alpha\in\mathcal{A}}E[\pi]-1}{\sup\limits_{\alpha\in\mathcal{A}}E[\pi]+1}$ is driven by the agents' payoff and the local topological configuration.

If π is a positive integer, $l=1$ and the Boolean game determine the interaction between agents. Theorem 3.1 and Theorem 3.2 were discovered by Bollobàs, Kozma, and Miklàs, (2008), and they are consistent with the results of Oliveira et al., (2014); Cho et al., (2013), and Schneider et al., (2011). If $l=1$, $t_c=0$, π is a positive integer and the Boolean game dominates the interaction between agents, this system satisfies the strong scale-free property and the robustness and vulnerability of this system are equal to the degenerative case of Theorem 3.1 and Theorem 3.2 (see Schneider et al., 2011). If $l<\infty$, then Theorem 3.1 and Theorem 3.2 are similar to the conclusions of Li and Chen (2003). In all, scholars insist that Theorem 3.1 and Theorem 3.2 are universal and scientific. Some interesting examples include *The Doctoral Dissertation Defense, Appropriate Scale Urbanization in China* and *Effective Strategies for Implementing the Chinese economic Reform* can be founded in the Appendix I of this article.

MATHEMATICAL DETAILS OF THE EVOLUTION PROCESS OF THE CAS

In general, whether an arbitrary system operates normally is determined by not only whether the most important agents in the system are connected, but also whether they are operating as usual. Because the system structure affects agents' strategies (see the Appendix A of this article), one must focus on the system's topological structure when the corresponding percolation is considered (Chen, 2016; Kranton,. 2020). It is important to first construct a precise mathematical model of stochastic process (Γ), an equivalent model of the original model, for describing this system's co-evolutionary process, outlined in Figures 2-4.

Based on Hypothesis 1, set $S=(S_1,S_2,...,S_l)$, and the n agents are allocated to l local-worlds in this system, which are labeled as $1_1,2_1,...,n_1,1_2,...,j_i,...,(n-1)_l,n$. Because of its complexity, our model is constructed based on a series correlation model (see the Appendix D of this article).

Evolution Process and the Law of Local Configuration for the Arbitrary Agent

Take a random variable $M_2(0,1)$ with density function $2x, 0<x<1$ in $[0,1]$. Let r_k be the i.i.d. random variables of $M_2(0,1)$, where $1\le k\le l\lfloor E[\pi]\rfloor n$, and π is the payoff at time t. Reordering these random variables in ascending sequence, obtain $0<R_1<...<R_{l\lfloor E[\pi]\rfloor n}$, set $W_{j_i}=R_{\lfloor E[\pi]\rfloor\left(\sum_{l=1}^{i-1}n_l+j\right)}, 1\le j\le n, 1\le i\le l, \sum_{j=1}^{n}\sum_{i=1}^{l}j_i=n$, where l is the number of local-worlds. Suppose that $W_0=0$, we denote $w_i:=W_{j_i}-W_{j_i-1}$.

Random complex networks $G_{\lfloor E[\pi]\rfloor}^{(n)}$ can be generated as follows: given R_k, we take $l\lfloor E[\pi]\rfloor n$ independent random variables $L_{j,i,r}$, where $1\le j\le n, 1\le i\le l, 1\le r\le\pi$, and $L_{j,i,r}$ is a random variable of uniform distribution at $R_{\lfloor E[\pi]\rfloor(j_i-1)+r}$. Each Agent j_i sends $\lfloor E[\pi]\rfloor$ edges to agent $t_{j,i,r}$, and each $t_{j,i,r}$ is unique such that $W_{t-1} < L_{j,i,r} < W_t$, where the labels i, j, r are the reordered label of agents. Reconsidering the structure of the CAS, one assumes that the i^{th} n_i agents are allocated in the i^{th} local-world, thus, $j_i = \sum_{l=1}^{i-1} n_l + j$.

Because $W_{t-1} < L_{j,i,r} < W_t$, it is of no distinction to take value for $L_{j,i,r}$ between $[0, W_{j_i}]$ and $[0, R_{m(j_i-1)+r}]$. Starting from W_{j_i} mentioned above, a random $t_{j,i,r}$ describes the vertex j_i send to it a link independently with probability of

$$P_1(t_{j,i,r} = t) = \begin{cases} w_t / W_{j_i}, & 1 \le t \le j_i \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Or, it would delete a link independently with probability of

$$P_2(t_{j,i,r} = t) = 1 - P(t_{j,i,r} = t) = \begin{cases} 1 - w_t / W_{j_i}, & 1 \le t \le j_i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

These two events, adding a game with a new agent or leaving an old game, would happen with probability $\Pr_1, \Pr_2$ respectively, which are called preferential attachment and preferential abandonment. Assuming that these six behaviors mentioned in Figure 4 happen with probability $q_1, q_2, ..., q_6$, take $\Pr_1 = \dfrac{q_2 + q_3 + q_5}{1 - q_1}$, $\Pr_2 = \dfrac{q_4 + q_6}{1 - q_1}$, q_1 describing mechanisms of preferential attachment and preferential abandonment and stable local topological configuration. The former two events satisfy the condition of Bernoulli's test coupled with binomial distribution. Furthermore, at each time, one of these two events would happen with probability $\Pr_1$ and $\Pr_2$.

Combining Eq. (1) and Eq. (2), vertex $t_{j,i,r} = t$ is selected and linked to vertex j_i independently with the following probability:

$$P(t_{j,i,r} = t) = \begin{cases} \dfrac{q_4 + q_6}{1 - q_1} + \dfrac{q_2 + q_3 - q_4 + q_5 - q_6}{1 - q_1} \dfrac{w_i}{W_{j_i}}, & 1 \le t \le j_i \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Furthermore, there are an average of $\dfrac{(q_2 + q_3 + q_5)n}{1 - q_1}$ links that would be added and $\dfrac{(q_4 + q_6)n}{1 - q_1}$ links that would be deleted. Scaling the number of all n agents to be 1 and assuming that the number of agents added to a new game are scaled to be x, the number of agents to be removed from a game must be equal to $1 - x$. Without loss of generality, set $x > 0.5$, system is growing in this time-scale, and not vice versa. As we can see, the number of r_k that falls in the interval $[1 - x, x]$ is independent on q_i and n satisfies the binomial distribution $B(l\lfloor E[\pi]\rfloor n, x^2 / 2)$ if and only if there are at least $\left\lfloor E[\pi]\left(\sum_{l=1}^{i-1} n_l + j\right) / 2\right\rfloor$ random variables r_k among all $l\lfloor E[\pi]\rfloor n$ random variables fall in interval

$[1-x,x]$. Therefore, we can conclude, for all $n^{1/2} \le \left(\sum_{l=1}^{i-1} n_l + j\right) \le n$, with very high probability $1-o(n^{-\epsilon})$ (henceforth called **wvhp**), where $\epsilon = 1/1000$

$$W_{j_i} = \sqrt{\left(\sum_{l=1}^{i-1} n_l + j\right)/2n}\left(1+O(n^{-1/4}\log n)\right) = \sqrt{j_i/n}\left(1+O(n^{-1/4}\log n)\right) \tag{4}$$

The distribution of w_{j_i} can be easily obtained for given $W_1, W_2, \ldots, W_{j_i-1}$. Set $W_{j_i} = y$ (similar hypothesis for $W_1, W_2, \ldots, W_{j_i-1}$), there are just $\left\lfloor E[\pi]\left(n-\left(\sum_{l=1}^{i-1} n_l + j\right)+1\right)/2\right\rfloor$ variables in r_k larger than y. Furthermore, these variables $r_1', \ldots, r'_{\left\lfloor E[\pi]\left(n-(\sum_{l=1}^{i-1} n_l + j)+1\right)/2\right\rfloor}$ taken from $W_1, W_2, \ldots, W_{j_i-1}, W_{j_i}$ must be independent from each other with density $2x/(1-y^2)$, where $y < x < 1$. In this sense, the conditional distribution of W_i can be determined by the $\lfloor E[\pi]\rfloor^{th}$ smallest of these $\left\lfloor E[\pi]\left(n-\left(\sum_{l=1}^{i-1} n_l + j\right)+1\right)/2\right\rfloor$ random variables coupled with the corresponding density function. If $\left(\sum_{l=1}^{i-1} n_l + j\right)/2$ is not close to 1 or n, and if y satisfies Eq. (4), it is easy to know that $r_{j_i'}$ is matched with the Poisson distribution with density function of $E[\pi]\left(n-\left(\sum_{l=1}^{i-1} n_l + j\right)+1\right)y/(1-y^2) \sim E[\pi]\sqrt{\left(\sum_{l=1}^{i-1} n_l + j\right)n}$ when it is close to y. Thus, for given W_{i-1}, the distribution of w satisfies

$$\frac{Z_{\lfloor E[\pi]\rfloor}}{2\lfloor E[\pi]\rfloor\sqrt{\left(\sum_{l=1}^{i-1} n_l + j\right)n/2}} \tag{5}$$

where, $Z_{\lfloor E[\pi]\rfloor}$ can be given by the sum of $\lfloor E[\pi]\rfloor$ variables of exponential distribution with an average of 1.

How the neighbors of agent $j_i \in V\left(G^{(n)}_{\lfloor E[\pi]\rfloor}\right) = [n]$ grow when the system is developing is important. According to Hypothesis 3, the arbitrary agent makes a decision according to Figure 4 by invoking information from both his historic strategies within memory length and the behaviors of other agents within the game radius (Mann, 2020). Meaning, the local configuration is changed because of agent behaviors.

It is known from Eq. (3) that an agent's strategy relies on two parameters: W_{j_i}, the first parameter, is decided by agent j_i, or denoted by the scale $\alpha(j_i) = \left(\sum_{l=1}^{i-1} n_l + j\right)/n$, which is used to describe the property of W_{j_i} in forthcoming sections. w_{j_i}, the second parameter, can be scaled in a more natural way by $x(j_i) = 2\lfloor E[\pi]\rfloor\sqrt{\left(\sum_{l=1}^{i-1} n_l + j\right)n}\,w_{j_i}$. In this sense, it is concluded that $\alpha(j_i)$ satisfies the uniform distribution on $[0,1]$ but $x(j_i)$ satisfies the distribution of $Z_{\lfloor E[\pi]\rfloor}$ (defined by Eq. (5)) that is independent of $\alpha(j_i)$, if an arbitrary agent j_i is selected.

There are two kinds of neighbors for agent j_i, one is the left neighbor if $j'_{i'} < j_i$, which means that agent j_i acts with an arbitrary agent $j'_{i'}$ positively. The other is called the right neighbor if $j'_{i'} > j_i$ meaning that agent j_i is linked by an arbitrary agent $j'_{i'}$ passively. Considering the r^{th} left neighbor $j'_{i'} = t_{j,i,r}$, for all $t \leq j_i$, one can have $P(j'_{i'} = t) = w_t / W_{j_i}$. Therefore, we only need to consider whether the case of j_i and $j'_{i'}$ are not close to 1 or n. That is, they are in interval $\left[n^{1/2}, n - n^{1/2}\right]$, recalling Eq.(4), we have

$$\begin{aligned} P(j'_{i'} = t) &= \frac{q_4 + q_6}{1 - q_1} + \frac{q_2 + q_3 - q_4 + q_5 - q_6}{1 - q_1} \times \frac{w_i}{W_{j'_{i'}}} \\ &\sim \frac{q_4 + q_6}{1 - q_1} + \frac{q_2 + q_3 - q_4 + q_5 - q_6}{1 - q_1} \times \frac{x(t)}{2\left\lfloor E[\pi]\right\rfloor \sqrt{\left(\sum_{l=1}^{i'-1} n_l + j'\right) t}} \end{aligned} \tag{6}$$

The parameters $\alpha(j'_{i'}) = \left(\sum_{l=1}^{i'-1} n_l + j'\right) / n$ and $x(j'_{i'})$ are important, for simplicity, one denotes $\alpha(j'_{i'})$ to α and $x(j_i)$ to x, and set $f_z(y) = y^{\lfloor E[\pi]\rfloor - 1} \exp(-y) / \left(\left\lfloor E[\pi]\right\rfloor - 1\right)!$ is the distribution density function of random variable $Z_{\lfloor E[\pi]\rfloor}$, where $y > 0$. It is known that for all not too small $\beta, \beta < \alpha$ and not too large $y > 0$, the probability of $\alpha(j'_{i'})$ standing in the interval $[\beta, \beta + d\beta]$ and $x(j_i)$ standing in the interval $[y, y + dy]$ should be

$$n\left\lfloor E[\pi]\right\rfloor d\beta f_z(y) dy \frac{y}{2\left\lfloor E[\pi]\right\rfloor \sqrt{\alpha n \beta n}} = \frac{y f_z(y)}{2\left\lfloor E[\pi]\right\rfloor \sqrt{\alpha\beta}} d\beta dy \tag{7}$$

In fact, there are $nd\beta$ agents with $\alpha(t)$ in this interval. As such, the quantity of the agents with $x(t)$ would be described, according to the definition of w_{j_i}, as the probability of product of $nd\beta f_z(y)dy$ and Eq.(6) and Eq.(7). Note that there are no $x = x(j_i)$ in Eq.(5).

Now, we analyze the right neighbors of agent j_i. Their number is a random variable and, in a fitness scale, the probability that agent j_i has a right neighbor with $\alpha(j'_{i'}) \in [\beta, \beta + d\beta]$ and $x(j'_{i'}) \in [y, y + dy]$ is

$$w_{j_i} / W_{j'_{i'}} \sim \frac{x(j_i)}{2\left\lfloor E[\pi]\right\rfloor \sqrt{\sum_{l=1}^{i-1}\left(n_l + j\right)\sum_{l'=1}^{i'-1}\left(n_{l'} + j'\right)}} = \frac{x(j_i)}{2\left\lfloor E[\pi]\right\rfloor \sqrt{j_i j'_{i'}}}$$

Set the interval is small enough, it is easy to see that in this event agent j_i will have a right neighbor with two parameters of $\alpha(j'_{i'})$ and $x(j'_{i'})$.

Because the degree of agent j_i is just equal to the sum of $\left\lfloor E[\pi]\right\rfloor$, so, for all $j_i \geq n^{1/2}$, one has

$$\mathbb{E}(d(j_i)) \sim \left|\frac{q_4+q_6}{1-q_1}E[\pi]\right| + \frac{q_2+q_3-q_4+q_5-q_6}{1-q_1} \times \frac{x(j_i)}{2\sqrt{\sum_{l=1}^{i-1} n_l + j}} \left(\sum_{j'_{i'}=j_i}^{n} \sum_{l'=1}^{i'-1} (n_{l'} + j')\right)$$
$$\sim \left|\frac{q_4+q_6}{1-q_1}E[\pi]\right| + \frac{q_2+q_3-q_4+q_5-q_6}{1-q_1} \times x(j_i)\frac{n}{\sum_{l=1}^{i-1} n_l + j - 1} \quad (8)$$

Evolution Process and the Law of Local Configuration for All Agents

It is easy to see that, for all conditions W , for an arbitrary agent with $d(j_i) \geq n^{\epsilon}$ must hold $\mathbb{E}(d(j_i)) \geq 2n^{\epsilon}$. The distribution of neighbors of an arbitrary single agent can be obtained according to the results mentioned above. However, there are many agents in this CAS. Because neighbors are selected randomly, the distribution of local configuration must be changed randomly. Another question arises: what is the distribution of local configuration of all agents? To determine this, the following model is constructed:

Given an arbitrary agent v in corresponding graph G and an arbitrary integer $k \geq 0$, set $\Gamma_k(v)$ expresses the set of agents whose distance from agent v is just k . Set initial agent is v_0 , and set $\Gamma_k = \Gamma_k(v_0)$. So, $\Gamma_0 = \{v_0\}$, but for $k = 1,2,\ldots$, set Γ_k consists of all of the agents in graph $G^{(n)}_{\lfloor E[\pi]\rfloor} \setminus (\Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_{k-1})$. For certain $\Gamma_0,\ldots,\Gamma_k$, one can obtain the distribution of Γ_{k+1} and their values of $x(t)$. Similarly, N_k is the set $N_k(v_0) = \Gamma_0 \cup \ldots \cup \Gamma_k$. They arrived from left of the agent, so they are not only in set Γ_k , but also the right neighbor of every agent $j'_{i'} \in \Gamma_{k-1}$. Alternatively, if they arrived from the right of the agent, they are the left neighbor. On the contrary, each agent may have arrived from two sides. In the research process, the cycle is omitted. Given $\Gamma_0,\ldots,\Gamma_k$ and an arbitrary agent $j'_{i'} \in \Gamma_{k-1}$ arriving from the right side, the value of corresponding $x(.)$ can be known until there are no neighbors left $t_{i',j',r}$ in Γ_{k-1} , where $1 \leq r \leq \pi$. Suppose that this set is not too large, then there must exist a conditional distribution of $t_{i',j',r}$. For every $j_i > j'_{i'}$, all events of agent j_i linking to agent $j'_{i'}$ are independent unless every agent can be the left neighbor of $j'_{i'}$, given all W . Therefore, the non-conditional distribution of the right neighbor can be obtained. In this sense, it is clear that certain left neighbors of agent $j'_{i'}$ would be in N_{k-1} . However, the other neighbors must be in N_{k-1} with a certain non-conditional distribution.

Although it consists of perceptual intuition, model (Γ) is difficult to resolve. As such, we constructed an equivalent stochastic process $(\tilde{\Gamma}) = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \ldots) v_0 \alpha(v) \in (0,1)$. Eac $Z_{\lfloor E[\pi]\rfloor}$ h generation $\tilde{\Gamma}_k$ consists of several limited vertexes v and the number is coupled with a positive number $l(v) \in \{\lfloor E[\pi]\rfloor - 1, \lfloor E[\pi]\rfloor\}$ and two real numbers and $x(v) > 0$. Starting from $\tilde{\Gamma}_0$, there is just a vertex in system, where $\alpha(v_0)$ is randomly selected in $[0,1]$ and $x(v_0)$ satisfies the distribution mentioned above. Given $\tilde{\Gamma}_0, \tilde{\Gamma}_1, \ldots$ and arbitrary vertex $v \in \tilde{\Gamma}_k$, the independent growth process of the next generation can be described as follows. Denote $\alpha(v)$ to α and $x(v)$ to x . First, v can increase the link to the "left next generation" of $l(v)$, where w with $l(w) = \lfloor E[\pi]\rfloor$ and $\beta = \alpha(w)$ and $y = x(w)$ would be selected according to Eq. (6). Then, v can increase its "right next generation"

w with a certain Poisson number and each of them have $l(w) = \lfloor E[\pi] \rfloor - 1$, v has this vertex w, in a relative small interval, with $\beta = \alpha(w)$ and $y = x(w)$ can be obtained by Eq. (7).

Γ_k is equivalent to $\tilde{\Gamma}_k$ (proven in the Appendix F of this article). Furthermore, the analytic solution of $\tilde{\Gamma}_k$ exists, such that one can master the property of the system by analyzing the law of $\tilde{\Gamma}_k$. Furthermore, how graph $G^{(n)}_{\lfloor E[\pi] \rfloor}$ would be transferred to G_p and G_c is of interest. To achieve that, a generalized stochastic process $\tilde{\Gamma}$ defined above is introduced. Note that $x(.)$ can be omitted, such that $\tilde{\Gamma}$ can be explained to one dimension but not two dimensions, which makes the analysis process more feasible.

Suppose that $0 \le p \le 1$ and $0 \le c \le 1$ are different fixed constants. In particular, two aspects of properties of this system should be analyzed: G_p with $c = 0, p = 1$ and G_c with $c = 1, p = 0$ (i.e., all agents with $j_i \ge cn$ are deleted). When $\beta > c$, the probability density of Eq. (6) can be replaced by 0 to adjust the stochastic process G_p. In this process, the probability density should not be emphasized because it is changed to another when agents are removed. Thus, there are at least $l(v)$ left neighbors, but not exactly the initial $l(v)$ left neighbors . Similarly, if $\alpha(v) > c$, then $\tilde{\Gamma}_0$ should be substituted into $\varnothing$.

As for the event of deleting an agent randomly from graph $G^{(n)}_{\lfloor E[\pi] \rfloor}$, stochastic process $(\tilde{\Gamma})$ should be changed as follows. For arbitrary $v \in \tilde{\Gamma}_k$, the set of potential next generation agents can be constructed according to the old rules. In fact, v in $\tilde{\Gamma}_{k+1}$ would select one agent from the potential next generation set as its next generation with probability p, i.e.,, $\tilde{\Gamma}_0$ would be changed to $\varnothing$ with probability $1 - p$.

For certain $v \in \tilde{\Gamma}_k$, the distribution of $x(v)$, $(\tilde{\Gamma})$ are relatively simple. However, if v is selected randomly, which makes the distribution of $x(v)$ and $(\tilde{\Gamma})$ more complex. As known, properties of $x(v)$ and $(\tilde{\Gamma})$ are more important to discovering system configuration evolutional law. So, we construct a stochastic process $\tilde{\Gamma}_{k+1}$ to describe how agent v arrives from the left or from the right. Because v is a next right generation, it is easy to see from Eq. (7) that $x(v)$ satisfies the partial distribution of Z_m with probability density function $xf_z(x) / \lfloor E[\pi] \rfloor$.

Thus, $(\tilde{\Gamma})^{p,c} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \ldots)$ is defined as follows: Each $\tilde{\Gamma}_k$ is a set consisting of limited neighbors of agent v. v is included where there is an integer $l(v) \in (0,1)$. Set $\Gamma_{0'} = \{v_0\}$, where $l(v_0) = \lfloor E[\pi] \rfloor, s(v_0) = 0$ and $\alpha(v_0)$ is selected uniformly in $[0,1]$. If $\alpha(v_0) > c$, $\tilde{\Gamma}_0 = \varnothing$. Otherwise, $\tilde{\Gamma}_0 = \tilde{\Gamma}'_0$ with probability p and the $\tilde{\Gamma}_0 = \varnothing$ with probability $1 - p$.

Given $\tilde{\Gamma}_k$, $\tilde{\Gamma}_{k+1}$ is constructed as follows. At every $v \in \tilde{\Gamma}_k$, suppose that v is independent on the potential next generation $l(v)$ determined by an integer $w \in \tilde{\Gamma}_{k+1}$ with probability of Eq. (7). Thus, $l(v) = \lfloor E[\pi] \rfloor$, $s(w) = 1$ and $\beta = \alpha(w)$ satisfies probability density distribution

$$\frac{p}{2\sqrt{\alpha\beta}} d\beta, \beta < \alpha < c \tag{9}$$

where, $\alpha = \alpha(v)$. For next right generation is considered, if $s(v) = 0$, a random variable $x = x(v)$ would be constructed according to the definition of $Z_{\lfloor E[\pi] \rfloor}$. If $s(v) = 1$, another random variable $x = x(v)$ would be constructed due to $Z_{\lfloor E[\pi] \rfloor}$. Then, a corresponding stochastic process, the Poisson number of the right generation $w \in \tilde{\Gamma}_{k+1}$ with $l(w) = \left\lfloor E[\pi] \right\rfloor - 1, s(w) = 0$ is constructed such that the right w selected with $\beta = \alpha(w)$ would be given in a relatively small interval with probability density

$$\frac{px}{2\sqrt{\alpha\beta}} d\beta \tag{10}$$

As proven in the Appendix F of this article, $(\tilde{\Gamma})^{p,c}$ is equivalent to $(\Gamma)^{p,c}$. Thus, one can know the property of $(\Gamma)^{p,c}$ by only analyzing $(\tilde{\Gamma})^{p,c}$. Removing all agents with $j_i > cn$ from $G^{(n)}_{\lfloor E[\pi] \rfloor}$ and retaining the other agents independently with probability $1-p$ could be implemented. If the initial agent is removed with uniform distribution in $(\Gamma)^{p,c}$, then we have $\Gamma_0 = \varnothing$.

For arbitrary agent j_i, its number of neighbors would be described as

$$\mathbb{E}[L(\alpha)] = c_1 j_i^{d_2} \frac{p}{2\sqrt{\alpha}} \int_{\alpha}^{\beta=c} \frac{1}{2\sqrt{\alpha}} \left(1 - \frac{(1-L(\beta))^{\lfloor E[\pi] \rfloor_{j_i}}}{(1+R(\beta))^{\lfloor E[\pi] \rfloor_{j_i}+1}} \right) d\beta \tag{11}$$

$$\mathbb{E}[R(\alpha)] = c_2 j_i^{d_1} \frac{p}{2\sqrt{\alpha}} \int_{0}^{\alpha=\beta} \frac{1}{2\sqrt{\alpha}} \left(1 - \frac{(1-L(\beta))^{\lfloor E[\pi] \rfloor_{j_i}} - 1}{(1+R(\beta))^{\lfloor E[\pi] \rfloor_{j_i}+1}} \right) d\beta \tag{12}$$

where, $d_1 = -1 - \dfrac{q_1 + q_2 + q_3 + q_5 + q_6}{1 + \delta_{out}(q_1 + q_2 - q_5)}$, and $d_2 = -1 - \dfrac{q_1 + q_2 + q_4 + q_5 + q_6}{1 + \delta_{in}(q_1 + q_2 - q_5)}$. The corresponding proof can be found in the Appendix D of this article.

RANDOM ATTACKS AND THE SYSTEM'S ROBUSTNESS

G_p, reflecting a system is attacked randomly, is defined as follows: By removing several agents as probability $1-p$ from $G^{(n)}_{\lfloor E[\pi] \rfloor}$, we obtain a generalized stochastic process $(\tilde{\Gamma})$.

For analytical convenience, define $E[\pi] \xrightarrow{def} \tilde{\pi}$. Fixed positive $p > 0$, and set $c = 0$. It is known that $G^{(n)}_{\lfloor \tilde{\pi} \rfloor}$ is obtained by removing agents randomly with probability $1-p$ and retaining the other agents in the system. Because $L(\alpha)$ and $R(\alpha)$ cannot be zero in the co-evolutionary process $\tilde{\Gamma}^{p,0}$ function $L_0(\alpha) = 0$ is introduced

$$R_0(\alpha) = \begin{cases} 1, & 0 < \alpha < \alpha_0 \\ 0, & \alpha_0 < \alpha \leq 1 \end{cases} \tag{13}$$

Set $(L_1, R_1) = \mathbf{F}((L_0, R_0))$, for arbitrary $\alpha > \alpha_0$, $L_1(\alpha) \geq 7p / 8\sqrt{\alpha_0 / \alpha}$ hold. The value of the other parts are denoted by $L_1 \geq 0$. Set $(L_2, R_2) = \mathbf{F}((L_1, R_1))$, for all $\alpha \leq \alpha_0$ within down-bound L_1 and $R_1 \geq 0$, it is concluded that

$$E(R_2(\alpha)) \geq d\left(\frac{n}{2}\right)^{-d_2} \frac{p}{2\sqrt{\alpha}} \int_{\beta=\alpha_0}^{1} \frac{7p\sqrt{\alpha_0}}{8\sqrt{\beta}} d\beta = d\left(\frac{n}{2}\right)^{-d_2} \frac{7p^2}{16} \log(1/\alpha_0) \tag{14}$$

Set $\alpha_0 = \exp(-16p^{-2}/7)$, for $0 < \alpha \leq \alpha_0$, we have $R_2(\alpha) \geq 1 = R_0(\alpha)$, $(L_2, R_2) \geq (L_0, R_0)$ and $(L, R) \geq (L_0, R_0)$ because there must exist a series (L_i, R_i) such that $(L_i, R_i) \geq (L_{i-2}, R_{i-2})$ holds. Because L, R have their own upper-boundary and lower-boundary, this event is converged into a certain point as long as both odd terms and even terms are considered. So, the probability $\sigma(p, 0)$ never dies off and $(\tilde{\Gamma})^{p,0}$ is positive everywhere. In fact, when $p \to 0$, we have

$$\sigma(p, 0) \geq \exp(-\Theta(-p^{-2})) \tag{15}$$

Denoting the rank of the first and second largest components of the graph as $L_1(G)$ and $L_2(G)$, respectively, we obtain Theorem 3.1, which is proven as follows:

Proof. Fix $0 < p \leq 1$, and set $c = 0$, for all $0 \leq k \leq n$, denote the agents' number of components with rank k in G_p is N_k (the number of agents of the component with rank k in graph G), and denote the rank of the removed components as 0. N_0 satisfies binomial distribution $N(n, (1-p))$. Denoting μ_k as the probability of event $\left|\bigcup_{t=0}^{\infty} \tilde{\Gamma}_t\right|$ happening in the stochastic process $\tilde{\Gamma}^{p,0}$, for all $0 \leq k < \infty$, we have $\sum_{k=0}^{\infty} \mu_k = 1 - \sigma(p, 0)$. For all $0 \leq k \leq n^{\epsilon}$,

$$\mathbb{E}(N_k) = n\mu_k + o(n^{1-\epsilon}) \tag{16}$$

As we can see, N_k is converged into its average. So, for all k in $0 \leq k \leq n^{\epsilon}$, we have **wvhp**

$$N_k = \mathbb{E}(N_k) + O(n^{2/3}) = n\mu_k + o(n^{1-\epsilon}) \tag{17}$$

Note that μ_k in-depends on n and $\sum_k \mu_k$ converges. We have $\mu_k \to 0$, which means **wvhp** that there exists a big component in graph G_p with $\sigma(p, 0)n + o(n)$ agents (i.e., all agents with $0 \leq k \leq n^{\epsilon}$ must only have a giant component beside them).

Once certain agents' neighbors are enlarged, this growth is extensive. We denote j_{i_0} as the smallest agent in G_p. Each agent with $|\Gamma_k(v)| \geq (\log n)^{10}$ would interact with agent j_{i_0} with probability $1 - o(1)$. For an arbitrary fixed agent v, event $\left|\bigcup_{k=0}^{\infty} \Gamma_k(v)\right|$ cannot be happening unless each $|\Gamma_k(v)|$ with k is very small. Invoking the results mentioned in Theorem C.1 of this article, the corresponding properties can be demonstrated.

It is known that $P(\alpha, l, s)$ is monotone decreasing with α, but not equal to 0 when $\alpha = 1$. So, it is bounded uniformly when α is far away 1. Similarly, for $\tilde{\Gamma} \geq 3$ and given probabilities $\alpha(v), l(v), s(v)$, it is induced that the probability of v with at least two propagable generations has its down-boundary. Suppose that v is propagable and the lower-boundary of the probability of v with at least two propagable generations can be driven by p_0. Due to the property of propagation of

v_0, there are at least two children generations of v_0 taken according to a probability p_0, which results in a branching process. Furthermore, this branching process is a misconverged exponent process. So, we have **wvhp** $\tilde{\Gamma}_{\log n} \geq (\log n)^{10}$, i.e., for each k, $\left|\bigcup_{j_i=0}^{k} \tilde{\Gamma}_{j_i}\right| < n^{\epsilon}$ holding, the probability of $\left|\bigcup_{j_i=0}^{k} \tilde{\Gamma}_{j_i}\right| < (\log n)^{10}$ is $o(1)$. So, the agents' number of G_p is $o(n)$ in the giant component.

Because $\sum_k \mu_k = 1 - \sigma(p,0)$, there exists an upper-boundary of $L_1(G_p)$ such that $\sigma(p,0)$ has a certain upper-boundary. Considering a small enough α_0, the agents with $\alpha \leq \alpha_0$ are called good agents and the ones with $\alpha > \alpha_0$ are called bad agents with a probability of 1. For agent v_0, it would have at least one child generation **wvhp**. Among the infinite generations of children, each child must have its left child with a positive probability.

For a set V that consists of some agents in $(\tilde{\Gamma})^{p,c}$, $\mathcal{R}(V)$ is the set of the right neighbors. Setting $\mathcal{R}^k(V) = \mathcal{R}(\mathcal{R}^{k-1}(V))$, $\mathcal{R}_\infty(V) = \bigcup_{k=0}^{\infty}(\mathcal{R}^{k-1}(V))$ and $\mathcal{R}_\infty^+(V) = \mathcal{R}_\infty(\mathcal{R}(V)) = \bigcup_{k=1}^{\infty}(\mathcal{R}^k(V))$ and considering Eq.(9), the average of $Z_{\lfloor\tilde{\pi}\rfloor}$ is $\lfloor\tilde{\pi}\rfloor + \Delta\lfloor\tilde{\pi}\rfloor$, for all $\alpha < \beta < 1$. So, arbitrary agent v with $\beta = \alpha(w)$ coupled with probability density $\rho_1(\alpha,\beta)d\beta$ in $\mathcal{R}(\{v\})$ with a small interval can be determined by

$$\rho_1(\alpha,\beta) = d\frac{p(\lfloor\tilde{\pi}\rfloor + s(v))}{2\sqrt{\alpha\beta}}(n/2)^{-d_1}d\beta \tag{18}$$

Because each agent w in $\mathcal{R}(\{v\})$ has $s(w) = 0$, i.e., for all $k \geq 1$, the probability density function $\rho_k(\alpha,\beta)$ of $\mathcal{R}^k(\{v\})$ satisfies

$$\sum_{k\geq 1}\rho_k(\alpha,\beta) \leq d\frac{p\lfloor\tilde{\pi}\rfloor}{\sqrt{\alpha\beta}}(n/2)^{-d_1} \tag{19}$$

As demonstrated in the Appendix G of this article, if $\alpha > \alpha_0 = \exp(-1/2\lfloor\tilde{\pi}\rfloor p)$, the boundary of the exponent described above can be obtained due to the condition $\beta \leq 1$.

Reconsidering the definition of arbitrary agents in $G_{\lfloor\tilde{\pi}\rfloor}^{(n)}$, the weight of v of $(\tilde{\Gamma})^{p,c}$ must be $1/\sqrt{\alpha(v)}$, and the weight of the set must be equal to the sum of all agents' weights. Supposing that v is a bad agent (i.e., $\alpha = \alpha(w) > \alpha_0$), we have

$$\begin{aligned}\mathbb{E}\left(w(\mathcal{R}_\infty^+(\{v\}))\right) &= d(n/2)^{-d_1}\int_{\beta=\alpha}^{1}\frac{1}{\sqrt{\beta}}\sum_{k\geq 1}\rho_k(\alpha,\beta)d\beta = d(n/2)^{-d_1}\int_{\beta=\alpha}^{1}\frac{p\lfloor\tilde{\pi}\rfloor}{\beta\sqrt{\alpha}}d\beta \\ &\leq p\lfloor\tilde{\pi}\rfloor\log(1/\alpha)w(v) \leq dw(v)n^{-d_1}/2^{1+d_1}\end{aligned} \tag{20}$$

Because $\mathcal{R}(\{v\}) = \{v\} \cup \mathcal{R}_{\infty}^{+}(\{v\})$, we have, $\mathbb{E}\left(w(\mathcal{R}_{\infty}^{+}(\{v\}))\right) \leq 3dw(v)n^{-d_1} / 2^{1+d_1}$. So, for the set including an arbitrary bad agent, it is easy to see that $\mathbb{E}\left(V(\mathcal{R}_{\infty}^{+}(\{V\}))\right) \leq 3dw(V)n^{-d_1} / 2^{1+d_1}$. $\mathcal{R}(V)$ is a random set that $\mathcal{L}(V)$ only consists of bad agents, where $\mathbf{1}_{\mathcal{R}(V)}$ is the corresponding indication function. In this sense, we have

$$\mathbb{E}\left(\mathbf{1}_{\mathcal{B}(V)} w\left(\mathcal{R}_{\infty}(V)\right) \mid V\right) \leq 3 \times \mathbf{1}_{\mathcal{R}(V)} dw(v)n^{-d_1} / 2^{1+d_1} \tag{21}$$

$\mathcal{L}(V)$ is the set of the generation of left children of V. Noted firstly, take integral to Eq. (9) from 0 to α_0 and one can easily determine that the probability that a certain children generation of agent V is "*good*" is $p\sqrt{\alpha_0 / \alpha(v)} = p\sqrt{\alpha_0 w(v)}$. So, it is concluded that

$$\mathbb{P}(\mathcal{L}(V) \textit{has a good agent}) \leq \tilde{\pi} p\sqrt{\alpha_0} w(V) \tag{22}$$

Now, we consider the bad agent in $\mathcal{L}(V)$. For each generation of left children for agent v with $\alpha(v) = \alpha > \alpha_0$, we take integral of $1 / \sqrt{\beta}$ then multiply Eq. (9). We have,

$$\begin{aligned}\mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}(\{v\}))} w(\mathcal{L}(\{v\}))\right) &\leq d(n/2)^{-d_1} l(v) \int_{\beta=\alpha_0}^{\alpha} \frac{1}{\sqrt{\beta}} \frac{p}{2\sqrt{\alpha\beta}} d\beta = l(v) \frac{dp \log(\alpha / \alpha_0) n^{-d_1}}{2^{1+d_1}\sqrt{\alpha}} \\ &\leq d\left|\tilde{\pi}\right| p \log(1/\alpha_0) w(v) n^{-d_1} / 2^{1+d_1} = dw(v)n^{-d_1} / 2^{1+d_1}\end{aligned} \tag{23}$$

We now have obtained all results. The probability that v_0 is a good agent is α_0, $\mathcal{L}_0 = \mathcal{R}_{\infty}(\{v_0\})$. Because $k \geq 1$, set $\mathcal{L}_k = \mathcal{R}_{\infty}(\mathcal{L}(\mathcal{L}_{k-1}))\mathcal{L}_k$ is the set of agents for all the children of v_0. So, if v_0 is bad but one of the children is good, there must exist a good agent in $\mathcal{L}(\mathcal{L}_k)$. It is possible for a certain bad agent to reach another good agent. Suppose that $\mathcal{L}_0, \ldots, \mathcal{L}_k$ consists of a bad agent, we obtain the following by combining Eq. (22)

$$\mathbb{P}\left(\mathcal{L}_k \text{ is bad agent but } \mathcal{L}(\mathcal{L}_k) \text{ is not} \mid \mathcal{L}_k\right) \leq d\left|\tilde{\pi}\right| p(n/2)^{-d_1} \sqrt{\alpha_0} \mathbf{1}_{\mathcal{R}(\mathcal{L}_k)} w(\mathcal{L}_k) \tag{24}$$

So

$$\mathbb{P}\left(\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_k \text{ is bad agent but } \mathcal{L}(\mathcal{L}_k) \text{is not} \mid \mathcal{L}_k\right) \leq \left|\tilde{\pi}\right| p\sqrt{\alpha_0} \mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_k)} w(\mathcal{L}_k)\right) \tag{25}$$

Similarly, to the event where v_0 has a good child, the boundary of $\sigma(p, 0)$ is

$$\sigma(p, 0) \leq \alpha_0 + \sum_{k=0}^{\infty} \left|\tilde{\pi}\right| p\sqrt{\alpha_0} \mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_k)} w(\mathcal{L}_k)\right) \tag{26}$$

Therefore,

$$\mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}_0)} w(\mathcal{L}_0)\right) \leq \frac{3}{2}\mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\{v_0\})} w(\{v_0\})\right) = \frac{3}{2^{1+d_1}} dn^{-d_1} \int_{\beta=\alpha_0}^{1} \frac{1}{\sqrt{\beta}} d\beta = \frac{3d}{2^{d_1}} n^{-d_1} \sqrt{\alpha_0} \tag{27}$$

Combining Eq. (15) and Eq. (26), and considering the definition of $\mathcal{L}_k$, for all $k \geq 1$, it is easy to see that

$$\mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_k)} w(\mathcal{L}_k)\right) \leq \frac{3}{8}\mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_{k-1})} w(\mathcal{L}_{k-1})\right) \tag{28}$$

Recalling Eq. (26), we obtain that

$$\begin{aligned}\sigma(p,0) &\leq \alpha_0 + \sum_{k=0}^{\infty} \tilde{\pi} p \sqrt{\alpha_0} \frac{3d}{2^{d_1}} n^{-d_1} \sum_{k=0}^{\infty} \sqrt{\alpha_0} (3/8)^k \mathbb{E}\left(\mathbf{1}_{\mathcal{R}(\mathcal{L}_0 \cup \ldots \cup \mathcal{L}_k)} w(\mathcal{L}_k)\right) \\ &\leq (1 + 5dn^{-d_1} \tilde{\pi} p / 2^{d_1})\alpha_0 \leq (1 + 5d\tilde{\pi} p / 8)\alpha_0 = (1 + 5d\tilde{\pi} p / 8)\exp(-1/2\tilde{\pi} p)\end{aligned} \tag{29}$$

As mentioned above, $(\sigma(p,0) + o(1))n$ is the up-boundary of the number of agents of the giant-component $\mathcal{L}_1(G_p)$ in G_p. Thus, the upper-boundary of Theorem 3.1 is demonstrated. Theorem 3.1 is proven.

INTENTIONAL ATTACK AND THE VULNERABILITY

We now discuss the case of the intentional attack (The percolation of a CAS with the Boolean game is described in the Appendix G of this article). First, the agents should be re-ordered according to payoff from large to small, and we obtain the following new order

$$\underbrace{1,2,\ldots,cn,}_{\text{core}} \; \underbrace{cn+1,\ldots,\tilde{c}n,}_{\text{intermediate asymptotic}} \; \underbrace{\tilde{c}n+1,\ldots,n}_{\text{periphery}} \tag{30}$$

We now analyze the properties of the intentional attack with probability β of $G_{\lfloor \tilde{\pi} \rfloor}^{(n)}$ (i.e., we remove all agents with $j_i \leq cn$ and obtain graph G_c). When $\beta \leq c$, we can implement the process of change G_p by setting the density function described in Eq. (3) to 0. In the process of changing, the density function would be changed because the cn strongest agents are removed.

We assume that $p = 1$ and consider the stochastic process $(\tilde{\Gamma})^{1,c}$ on $0 \leq c \leq 1$. We denote the propagation probability of the initial agent v_0 in the stochastic process $(\tilde{\Gamma})^{1,c}$ as $\theta(c) = \sigma(1,c)$. Because $(\tilde{\Gamma})^{1,c}$ consists of an infinite number of agents, we denote μ_k relying on c as the probability of k agents selected in the stochastic process. Based on the definition of μ_k, we have

$$\sum_{k=0}^{\infty} \mu_{k,c} = 1 - \theta(c) \tag{31}$$

For $c_1 < c_2$, $(\tilde{\Gamma})^{1,c_2}$ can be defined by removing all agents v with $c_1 < \alpha(v) \leq c_2$ from $(\tilde{\Gamma})^{1,c_1}$. $(\tilde{\Gamma})^{1,c_1}$ and $(\tilde{\Gamma})^{1,c_2}$ can be embedded together perfectly. So, $\theta(c)$ gradually decreases with c (demonstrated in the Appendix H of this article).

$\theta(c)$ is decreasing gradually and continuously, and that the critical value $c_0 := \inf c : \theta(c) = 0$ satisfies $0 < c_0 < 1$ when $\theta(0) = 1$ and $\theta(1) = 0$. Furthermore, $\theta(c_0) = 0$.

For $0 < c < c_0$, we have $\theta(c) > 0$, the probabilities of left propagation and right propagation of Eq.(11) and Eq.(12), $L_c(\alpha)$ and $R_c(\alpha)$, are all non-zero. Furthermore, because $\theta(c) \to 0$ when $c \to 0$, so, $L_c(\alpha)$ and $R_c(\alpha)$ all tend toward 0 when c is increased to c_0. Take limits of $L_c(\alpha)$ and $R_c(\alpha)$, and we have

$$\inf_{\alpha\in\mathcal{A}} L(\alpha) \leq \tilde{L}(\alpha) = \frac{1}{2^{d_1+1}\sqrt{\alpha}} dn^{-d_1} \int_{\beta=c}^{\alpha} \frac{1}{\sqrt{\beta}} \Big(\big\lfloor \tilde{\pi} \big\rfloor \tilde{L}(\beta) + (\big\lfloor \tilde{\pi} \big\rfloor + 1)\tilde{R}(\beta) \Big) d\beta \leq \sup_{\alpha\in\mathcal{A}} L(\alpha) \tag{32}$$

$$\inf_{\alpha\in\mathcal{A}} R(\alpha) \leq \tilde{R}(\alpha) = \frac{1}{2^{d_2+1}\sqrt{\alpha}} dn^{-d_2} \int_{\beta=\alpha}^{1} \frac{1}{\sqrt{\beta}} \Big((\big\lfloor \tilde{\pi} \big\rfloor - 1)\tilde{L}(\beta) + \big\lfloor \tilde{\pi} \big\rfloor \tilde{R}(\beta) \Big) d\beta \leq \sup_{\alpha\in\mathcal{A}} R(\alpha) \tag{33}$$

According to Eq. (32)-(33), $L_c(\alpha)$ and $R_c(\alpha)$ are transformed by the deterministic terms such that they are all resolved easily. Furthermore, they have a corresponding non-zero solution if and only if $c = \dfrac{\inf_{\alpha\in\mathcal{A}} \tilde{\pi} - 1}{\sup_{\alpha\in\mathcal{A}} \tilde{\pi} + 1} (n/2)^{d_2 - d_1} = \dfrac{q_3 - q_4}{1 + \delta_{in}(q_1 + q_2 - q_5)} \dfrac{\inf_{\alpha\in\mathcal{A}} \tilde{\pi} - 1}{\sup_{\alpha\in\mathcal{A}} \tilde{\pi} + 1}$.

As analyzed above, these equations have a non-zero solution if $c = c_0$

$$c_0 = \frac{q_3 - q_4}{1 + \delta_{in}(q_1 + q_2 - q_5)} \frac{\inf_{\alpha\in\mathcal{A}} \tilde{\pi} - 1}{\sup_{\alpha\in\mathcal{A}} \tilde{\pi} + 1} \tag{34}$$

It is concluded that the value of $\theta(c) = \sigma(1, c)$ is 0 if the probability of intentional attack $c \geq c_0$; otherwise, the value of $\theta(c) = \sigma(1, c)$ is positive. Also, if the number of agents in a component is smaller than $n^{\epsilon/2}$, the component is considered to be small. So, we have

$$N_{k,c} = \mu_{k,c} n + o(n^{1-\epsilon}) \tag{35}$$

For $0 \leq k \leq n^{\epsilon}$, we know that $N_{k,c}(n)$ is the number of agents of component G_c with rank k. For graph G_c, it is concluded **wvhp** that there are $(1 - \theta(c) + o(1))n$ agents in the small component and $(\theta(c) + o(1))n$ agents in the large component. When $c < c_0$, there only exists one large component in graph , as seen in Theorem 3.2, which is demonstrated as follows.

Proof. Set $\gamma = \gamma(n)$ is a function that slowly tends toward 0, supposing that we take $\gamma = n^{-\epsilon/3}$, and set $n' = \lfloor (1-\gamma)n \rfloor$. Recalling the initial definition of the stochastic process $G^{(n)}_{\lfloor \tilde{\pi} \rfloor}$, and considering the sequences embedded $G^{(t)}_{\lfloor \tilde{\pi} \rfloor}$, where $t = n', n'+1, ..., n$ and the corresponding abbreviated version H_t is obtained by deleting all agents with $j_i \leq cn$ in $G^{(n)}_{\lfloor \tilde{\pi} \rfloor}$ (i.e., $H_n = G_c$).

Because $\gamma \to 0$, for a large enough n, we have $c' = cn / n' < c_0$. Based on Eqs. (32) and (33), we replace n, c by n', c', and we see with **wvhp** that there are $(\theta(c') + o(1))n'$ agents G_c standing in the giant-component in graph $H_{n'}$. Because θ is continuous, the number should be equal to $(\theta(c) + o(1))n$. If a certain agent $1 \le j_i \le n$ stands in the giant-component $H_{n'}$, the agent is considered to be very important. Note that there are $(\theta(c) + o(1))n$ important agents in the system. Similarly, if a component of H_t consists of an important agent, where $n' \le t \le n$, then this component is also considered to be important. Also, note that the important agents cannot change with time but remain constant. However, when H_t transitions into H_{t+1}, there are one or more important agents in H_t that can link with another or more important agent in H_t. So, the number of important components would decrease with time. The objective of this paper is to demonstrate that there are same important components when $H_n = G_c$. In other words, when this important component consists of $(\theta(c) + o(1))n$ important agents, (i.e., all the $o(n)$ agents in G_c are in this component) the proof is finished.

Set A and B are different components with a and b agents in H_t, and set $d(A)$ is the sum of the number of agents with $j_i \in A$. Then, when $G^{t+1}_{\lfloor\tilde{\pi}\rfloor}$ is tended to infinity, the probability that a new link is constructed from a new agent to an arbitrary agent in A is $d(A) / (2\left\lfloor\tilde{\pi}\right\rfloor t + 1)$. Similarly, the second link is constructed from this new agent to B with probability $\dfrac{d(A)d(B)}{(2\left\lfloor\tilde{\pi}\right\rfloor t + 1)(2\left\lfloor\tilde{\pi}\right\rfloor t + 3)}$ when $t \sim n$, $d(A) \ge a$ and $d(B) \ge b$. So, for some positive constants of a and b, the probability of a second link occurring is at least $\kappa ab / n^2$. For $n' \le t \le t$, and set $\mathcal{L}_2(t)$ is the sum $|A||B|$ of different important component pairs $\{A, B\}$ in H_t. If $\mathcal{L}_2(t) \ge (\theta(c)n)^2 / 5$, then the components in H_t are connected together in H_{t+1} with probability $\kappa' = \theta(c)^2 \kappa / 5$. For this reason, the number of components in H_{t+1} is much smaller than the number in H_t. Because there are at least $n^{1-\epsilon/2}$ important components in $H_{n'}$, there are at most $n^{1-\epsilon/2}$ values coupled with t. For several t_0 with $t_0 \le n' + 2n^{1-\epsilon/2} / \kappa'$, we have **wvhp** that $\mathcal{L}_2(t) < (\theta(c)n)^2 / 5$. Thus, there is an important component C with a rank of magnitude larger than or equal to $\theta(c)n / 2$ in H_{t_0}. Furthermore, for another important component fixed in H_{t_0} and several constants with $\kappa' > 0$, it is easy to see that, in the other t ($t_0 \le t \le n$) steps, the probability that A and C can be connected is at least $\kappa |A||C| / n^2 \ge \kappa'' |A| / n\kappa'' n^{\epsilon/2-1}$. Because the probability that A and C can be connected in the remaining $n - t \sim n^{1-\epsilon/3}$ steps at least is $1 - o(n^{-2})$ all components in H_t can be connected together in H_n **wvhp**. In this sense, proof of Theorem 3.2 is finished.

DISCUSSION AND CONCLUSION

Our analysis shows that, if the system is attacked randomly, there are at least two large components that keep the system connected. Furthermore, when the system is attacked randomly, even if a few agents remain in the system, there must be two giant-components for the system to operate normally. The minimum of the rank of the biggest giant-component is $n\exp(\Theta(1 - p^2))$, and the maximum of the rank of the biggest giant-component should be $(1 + 5d\left\lfloor\tilde{\pi}\right\rfloor p / 8)n\exp(-1 / 2\left\lfloor\tilde{\pi}\right\rfloor p)$, which is driven by the average of the agent's payoff $\tilde{\pi}$. The rank of the second biggest giant-component is

$o(n)$. So, when the system is attacked randomly, the system is robust. That is, even if almost all agents of the system exit, the system remains strongly connected and does not collapse.

$\lambda(p)$ is a non-linear correlation with random attack probability p and agents' payoff π. Furthermore, $\lambda(p)$ is monotonously increasing with p, the upper bound of $\lambda(p)$ is affected by the property of the payoff of the agents, which is explained in Figure 5.

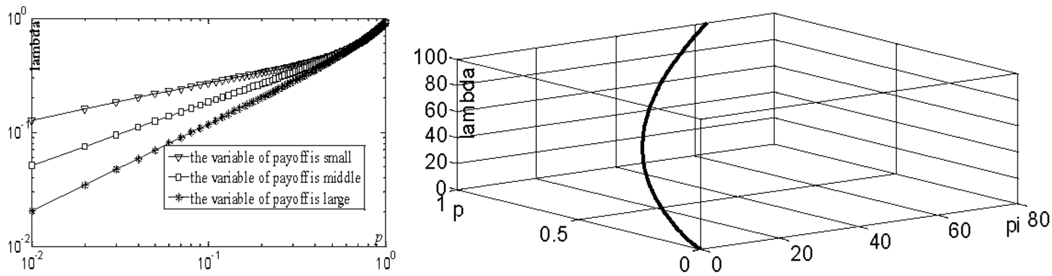
However, the corresponding difference between the $\lambda(p)$ sharply decreases as p increases. As shown in Figure 5, the payoff distribution also relies on the distribution of agents' type, there is a significant negative nonlinear correlation between $\lambda(p)$ and the average of payoff.

Our analysis also shows, when the system is attacked intentionally, the system's property is much more complex. The complexity comes from critical probability c_0. If the deleting probability is smaller than c_0, the system is connected by two giant components. If the deleting probability is larger than c_0, there is no existing giant component that makes the system connected, and the system's function is damaged. If the deleting probability is equal to c_0, the system is critical, i.e., the system is switched randomly between robustness and vulnerability. The size of the largest component relies on the property of $\theta(c)$. Unfortunately, the property is unknown, although $\theta(c)$ is decreasing gradually and continuously. The critical value $c_0 := \inf c : \theta(c) = 0$ satisfies $0 < c_0 < 1$ if $\theta(0) = 1$ and $\theta(1) = 0$, furthermore, $\theta(c_0) = 0$. Moreover, because $0 < c < c_0$, we have $\theta(c) > 0$ and $\theta(c) \to 0$ if $c \to 0$. Therefore, the number of left neighbors and right neighbors tend toward 0 if c is increased to c_0. Furthermore, there is a corresponding non-zero solution if, and only if,

$$c = \frac{\inf_{a \in \mathcal{A}} \tilde{\pi} - 1}{\sup_{a \in \mathcal{A}} \tilde{\pi} + 1} \left(\frac{n}{n} \right)^{d_2 - d_1} = \frac{q_3 - q_4}{1 + \delta_{in}(q_1 + q_2 - q_5)} \frac{\inf_{a \in \mathcal{A}} \tilde{\pi} - 1}{\sup_{a \in \mathcal{A}} \tilde{\pi} + 1}.$$

It is concluded that the critical probabilities p_0 and c_0 are dependent on the property of the collective payoff $\tilde{\pi}$. In fact, $\tilde{\pi}$ is determined by the agent's type τ and corresponding distribution, the transitory local configuration of the agent in random complex networks ω, and noise β. However, agents would change their property because of interaction, which makes the agent's type τ dynamical, whose distribution relies on how the in-homogenous agents are distributed, as well as the inhomogeneous behaviors. The local configuration of agent ω decides the interactive property (i.e., a cooperative game or a non-cooperative one). Because agents' optimal payoff satisfies certain invariable distribution with time t, and the payoff is monotonously increasing with system size N

Figure 5. The property of up-bound of $\lambda(p)$



but bounded no matter how large the system is, we can see that $\lim_{N\to\infty} \frac{\inf_{a\in\mathcal{A}} \tilde{\pi} - 1}{\sup_{a\in\mathcal{A}} \tilde{\pi} + 1} \approx 1$ such that $c_0 = \frac{q_3 - q_4}{1 + \delta_{in}(q_1 + q_2 - q_5)} o(1)$ with $N \to \infty$.

According to Zheng, et.al., (2012) agents' dynamic behavior must satisfy an invariable distribution $\mu^{\beta,N}\left(\omega \mid \Omega_\alpha^N\right)$ driven by noise β and local configuration ω. It is important to note how behavior and configuration affect the percolation probability p_0 and c_0. Furthermore, $\mu = o\left(\exp(x^2 / \beta^N) - \frac{\exp(x/\beta)}{\beta^{N-1}x}\right)$ and $x = N(q_1 + q_2 + q_3 - q_4 + q_5 - q_6)\pi$. It is concluded that, if the system is attacked intentionally, the critical probability would change, as shown as Figure 6.

Figure 6 describes how the critical probability c_0 changes according to agents' behavior and noise, where $0 \le x \le 8$ and $0 \le \beta \le 8$. There exists a tipping point such that critical probability c_0 changes sharply if the strategy and noise changes weakly. c_0 grows slowly if all of the resources and noise are smaller than this value, but c_0 increases quickly if they are above this value. The approximate value of the corresponding critical value is $\beta_c \approx 0.45, x \approx 0.52$.

Considered the system's behavior, we identified three behaviors in the system evolution process: agents optimize the flows in the system by adjusting their behavior in a stable structure; growth by adding a new game with other agents or simply adding other agents; or collapse by removing old game relationships or exiting the system. Set $q_1 = \lambda_1$, $q_2 + q_3 + q_5 = \lambda_2$ and $q_4 + q_6 = \lambda_3$ respectively describe a stable system, growth, and collapse. If $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$, and π is a positive integer, this is a BA model, and the percolation probability is similar to Theorem 3.1 and Theorem 3.2. When $\lambda_1 = 1$, and $\lambda_2 = \lambda_3 = 0$, this is an ER model. It satisfies Poisson distribution, the percolation probability is $1/(\kappa - 1)$, and κ is determined by the degree k and its probability p and the distribution $\varphi(k)$. That is, $\sum (k-1)p\varphi(k) = p(\kappa - 1)$, which is identical to the classic result. When $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$, the system is collapsed. Real-world examples of such systems include collapsing industries which die due to recessions. We randomly selected the value of q_i, and considered the critical probability of intentional attack driven by λ_i (see Figure 7).

Figure 6. $c_0(\beta, x)$

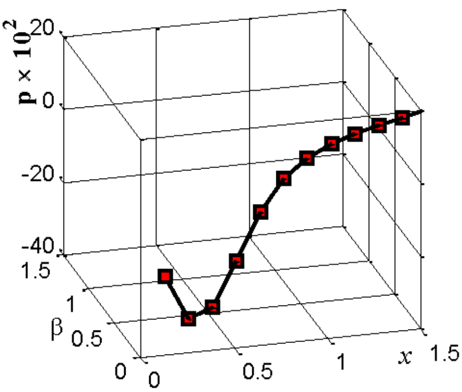
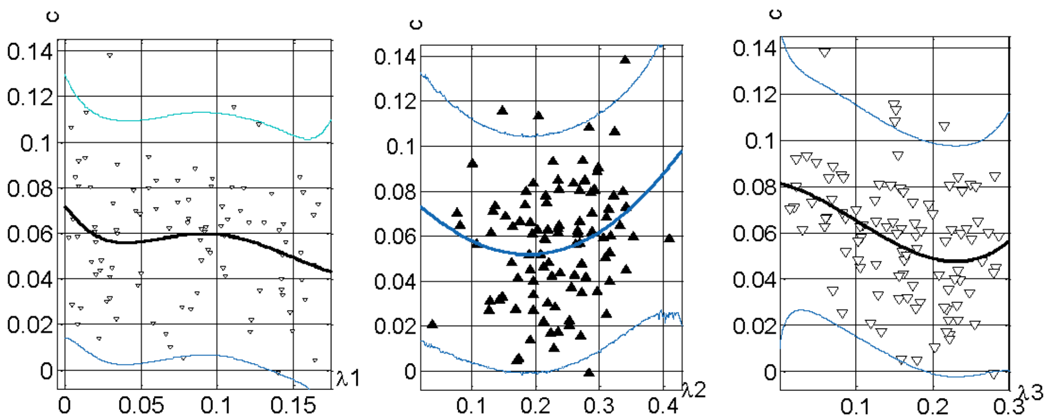


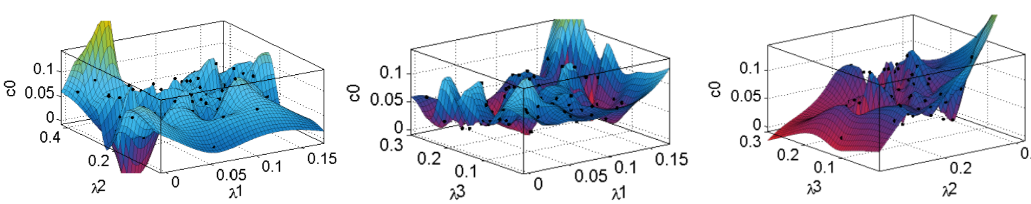
Figure 7. The correlation between system behavior and critical probability under intentional attack



The joint effect of two behaviors onto the critical probability of intentional attack is specified in Figure 8.

Obviously, an arbitrary agent in the system can only interact with some agents. The game radius r is introduced to describe the scale of indirect playing. r_i describes how many agents' information can be collected by an arbitrary agent i. In addition, any agent in reality cannot remember all information regarding past interactions, this property of limited remember is described by order parameter of remember length L. If the game radius is infinite, any agent in this system can interact with each other with perfect information, which has been studied (see Bab &Brafman, 2008; Chen, 2011; Ahn et al., 2008; Li et. al., 2010; Fuentes, Gerig, &Vicente, 2010; Kumar, 2010; Shao-Chiu Juan et.al., 2017; and Novičenko et. al., 2018). If agents can remember all past interactions, this means that the agent is very intelligent, this has been studied by Zheng, et.al. (2010; 2012). If agents can only remember little information, our result can be degenerated to what Sikder, Smith, et al., (2020) have proven. In this paper, we focused on how the critical percolation probability p_0 and c_0 are affected by the game radius r and agents' memory of past games. If $l = 1$, $r = 1, L = \infty$ and the Boolean game plays a dominant role, this system degenerates into a scale-free random complex network, which is also robust and vulnerable when the system is attacked randomly and intentionally, respectively (see Albert et al., 2000; Misra et al., 2010; Wang & Rong, 2009; Derzhko, 2004; Bollobàs et al., 2008; Bollobas & Riordan, 2007; Bowles, Baxter et al., 2011; Xiao Zang, et. al., 2020; Viktor Novičenko & Irmantas Ratas, 2018). However, the laws the percolation critical probability p_0 and c_0 satisfy should be further researched if $l \geq 2, L = M < \infty$ and $2 \leq r \leq \bar{M} < \infty$.

Figure 8. The correlation between system joint behavior and critical probability under intentional attack



However, there are too many conditions that were posed strictly in this paper, the results are not perfect. We would loosen the constraint conditions, such that the irrational behaviors are considered step by step in additional studies.

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COMPLIANCE WITH ETHICAL STANDARDS

Conflict of interest The authors declare that they have no competing interests.

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