International Journal of Chemoinformatics and Chemical Engineering
Volume 8 • Issue 2 • July-December 2019

On the Structure of Parallelohedrons of Higher Dimension:

Hilbert's 18th Problem

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ABSTRACT

For more than 100 years in science, many researchers, when trying to solve Hilbert's 18th problem of constructing n-dimensional space, used the principles of the Delaunay geometric theory. In this paper, as a result of a careful analysis of the work in this direction, it is shown that the principles of the Delaunay theory are erroneous. They do not take into account the features of figures of higher dimensionality, do not agree with modern advances in the physics of the structure of matter, and lead to erroneous results. A new approach to solving the 18th Hilbert problem, based on modern knowledge in the field of the structure of matter and the geometric properties of figures of higher dimension, is proposed. The basis of the new approach to solving the 18th Hilbert problem is the theory developed by the author on polytopic prismahedrons.

KEYWORDS

Dimension, Nanoworld, Parallelohedron, Polytope, Space, Stereohedron, Symmetry, Translation

INTRODUCTION

In 1885, the book of Efgraf Fedorov "Began the doctrine of figures" (Fedorov, 1885) went out of print. It classifies two - and three - dimensional figures that can fill the Euclidean plane and three - dimensional space adjacent to each other along whole edges and, accordingly, flat faces without gaps. These figures were called parallelogons and, correspondingly, parallelohedra, due to the fact that their sides are pairwise parallel. It gives the concept of stereohedrons as equal figures that make up a parallelohedron together. In 1889 and in 1891 the following works of Fedorov, "Symmetry of finite figures" and "Symmetry of regular systems of figures" (Fedorov, 1889, 1891), went out of print. In these works, Fedorov deduced 230 spatial groups, i.e. 230 unique geometric laws of arrangement of elementary particles of crystalline structures in three - dimensional Euclidean space and 17 two - dimensional groups of arrangement of elementary particles in the Euclidean plane. In 1891, Schönflis's book "Crystal Systems and the Structure of Crystals" was also published (Schönflis, 1891). Her author has repeatedly quoted Fedorov, pointing to his primacy in a number of questions on theoretical crystallography. Fedorov's extensive correspondence with Schönflis (Shafranovsky,

DOI: 10.4018/IJCCE.20190701.oa2

1963) has been preserved, from which the leading role of Fedorov in the derivation of 230 spatial symmetry groups follows. The studies were reflected in a report by David Hilbert (Hilbert, 1901) at the International Congress of Mathematicians in Paris, in which a number of mathematical problems were formulated that determined the development of mathematics for the next century. One of these problems (the eighteenth) raised the question of constructing spaces of higher dimension from congruent polyhedrons. The report expressed the desire to investigate the question of constructing spaces of higher dimension from congruent figures (Aleksandrov, 1969). The answer to this request was a series of researchers' work in subsequent years. However, many of them tried to consider spaces of higher dimension and figures of higher dimension by mechanically extending to a space of higher dimension ideas about three - dimensional Euclidean space without taking into account the essential properties and features of spaces of higher dimension (Minkowski, 1911; Delaunay, 1929, 1937, 1961; Delaunay, & Sandakova,1961; Voronoi, 1952; Alexandrov, 1934, 1954; Venkov, 1954, 1959). Although back in 1880, the work of the American mathematician Stringham appeared on figures in n - dimensional space (Stringham, 1880). This work was apparently unknown to these researchers, as well as the work of Stott from Amsterdam (Stott, 1900, 1910). Much later, the works of Coxeter and Grünbaum, widely known at present, on figures of higher dimensions appeared (Coxeter, 1963; Grünbaum, 1967). From the analysis of geometric figures of higher dimensions, it would be necessary to begin the solution of the question of constructing spaces of higher dimensions with the help of these figures. That is exactly what the ingenious crystallographer Fedorov did when he studied the question of constructing three - dimensional space with the help of three - dimensional figures. However, this did not happen. As a result, mechanically extending Fedorov's teaching on parallelohedra in three - dimensional spaces to the four - dimensional space, Delaunay in 1929 systematized, as he believed, four-dimensional parallelohedons (Delaunay, 1929). Moreover, no evidence was given for any of these 51 parallelohedrons that they have dimension 4. Later, a student of Delaunay Shtogrin, as part of the teachings of Delaunay, discovered the existence of another figure, additionally included in the classification of Delaunay. It was believed that this figure is also a parallelogon with dimension 4 (Shtogrin, 1973). In this chapter, it will be shown that, as a result of neglecting the necessary conditions for the existence of polytopes of higher dimension, none of the Delaunay classification figures, as well as the figure proposed by Shtogrin, are polytopes of higher dimension, i.e. due to this, they do not have, as it was supposed, a dimension equal to 4.

In the study of the structure of substances in recent decades, phenomena were discovered that were unknown at the time of David Hilbert. These include spontaneous zooming (scaling processes), for example, during second - order phase transitions (Landau, 1937; Kadanoff, 1966; Wilson, 1971); discovery of crystals with the absence of translational symmetry in three -dimensional space (Shehtman, et al., 1984); detection of the fractal nature of matter (Mandelbrott, 1982); hierarchical filling of space with polytopes of higher dimension (Zhizhin, 2012); the highest dimension of most molecules of chemical compounds (Zhizhin, 2018).

This forces us to expand the formulation of the question of constructing n - dimensional spaces, posed by David Hilbert in 1900, and to abandon the methods of considering the construction of spaces by geometric figures that do not take into account the accumulated discoveries in the physics of the structure of substances. Here we should take into account the new paradigm of the discrete world built in the latest works of Zhizhin (Zhizhin, 2018, 2019 a, b), in which the theory of polytopic prismahedrons is developed. They are the products of polytopes and, as a result, have several families of parallel edges that can ensure the filling of a space of higher dimension without gaps face to face.

The solution of Hilbert's 18th problem of constructing spaces of higher dimension using congruent figures is currently becoming especially relevant in connection with the established fact about the multidimensionality of the nanoworld (Zhizhin, 2013, 2014, 2018; Shevchenko, Zhizhin, & Mackay, 2013; Zhizhin, & Dudea, 2016). This, along with the theoretical mathematical value, gives the 18th Hilbert problem technological and, in the future, practical value (Zhizhin, 2020).

THE MAIN PROVISIONS OF THE DELAUNAY THEORY

Starting from 1929, Delaunay's theory has been stated more than once in various publications. Take as an example the presentation of this theory in the work of Shtogrin (1973). The paragraph "Basic concepts, definitions and tasks" says "The subject of our research will be the partitioning of n-dimensional Euclidean space into bodies - polyhedrons. The location of bodies in space at which they pairwise have no internal points is called packing. The location of bodies in space at where each point in space belongs to at least one of these bodies, is called a covering. A partition is a location of bodies in space that is both packing and covering. of interior points and cover the entire area. If the bodies in decomposition of convex polyhedrons that are. "

In these first lines, much remains uncertain. First of all, what are bodies in the space of higher dimension. If we follow Fedorov, who investigated the partition of three - dimensional space, then it would be necessary to first determine what kind of bodies, their properties and conditions of existence. This is nothing in the teaching of Delaunay.

The followers of Delaunay's teachings do not seem to know that a prerequisite for the existence of a polytope of higher dimension is the satisfaction of its Euler – Poincare equation. Until today, the Euler - Poincare equation has not been used in the presentation of Delaunay's theory by his followers. And this is understandable, since this equation would destroy the entire Delaunay theory.

Secondly, in Delaunay's theory, it remains unclear what to consider internal points of the body. This question is trivial when considering problems on the plane and in three - dimensional space. However, when moving into a space of higher dimension, this question becomes non - trivial. Therefore, back in 1967, Grünbaum introduced the concept of the boundary complex of a polytope of higher dimension, which includes many elements of different dimensions (Grünbaum, 1967). The ideas about the internal points of higher - dimensional polytopes are examined in detail in the author's monograph (Zhizhin, 2019 a). The issues of belonging of some elements of a certain dimension in a polytope of higher dimension to other elements of another dimension are also nontrivial. These issues are discussed in detail in this monograph (Zhizhin, 2019 a).

The division of space into polyhedrons Delaunay calls normal if these polyhedrons are adjacent along integer (n - 1) - dimensional faces (Shtogrin, 1973). It should be said here that this position again follows from representations of spaces of dimension 2 or 3. In these spaces, such a condition is equivalent to the absence of gaps between the elements of the "face to face" partition. However, in the space of higher dimension, the concept of "face to face" becomes uncertain, since a polytope of higher dimension has many faces of different dimensions. And the assertion that there will be no gaps between the elements of dividing a space of higher dimension is absolutely only necessary when they are adjacent along facets (i.e., (n - 1) - dimensional faces). The proof of Delaunay (1937) that polytopes in n - dimensional space either do not enter into each other or are adjacent along faces of dimension n -1 (facets) is not correct, since the concepts of three - dimensional space are used when considering n - dimensional space. The statement (Delaunay, 1937) that the vertices of the polytopes L_1 and L_2, if they lie on opposite sides of the chordal (n - 1) - dimensional "plane" of the balls (L_1) and (L_2) described around L_1 and L_2, has not been proved. We cite this statement from the work of Delaunay (1937).

"If the balls (L_1) and (L_2) intersect, then the vertices of the polytope L_1 lie on the same hat of the ball (L_1), which does not lie inside the ball (L_2), since the ball (L_2) is empty. Similarly, the vertices of the polytope L_2 lie on that cap of the ball (L_2), which does not lie inside the ball (L_1), since the ball (L_1) is empty, therefore, the vertices of the polytopes L_1 and L_2 lie on different sides of the chordal (n - 1) - dimensional "plane" of these balls, and therefore, the polytopes L_1 and L_2 themselves lie on opposite sides of this plane."

Obviously, if the polytopes L_1 and L_2 do not enter one into the other, but have common elements, then the remaining elements lie on opposite sides of the common elements. But why these common elements must have dimension *n* - 1 remained unproven. Polytopes in *n* - dimensional space can have common vertices, edges, common elements of any dimension from 0 to *n* - 1. Facets have no exclusivity in this regard.

We quote further the paragraph from the mentioned paragraph of Shtogrin's work. "Venkov, unlike Delaunay, calls a partition normal if it is a Dirichlet - Voronoi partition, and then it is normal in the sense of Delaunay. A partition of a space into polyhedrons is called a Dirichlet -Voronoi partition, if you can choose a point - center in each separate polyhedron of a partition action, such that this polyhedron of the partition is the Dirichlet - Voronoi region of its center of action with respect to all other centers of action, i.e. each polyhedron is a collection of points of space that are located b others like to the center of action than all other centers of action. " Note that this paragraph has never said that space has a higher dimension, although, judging by the previous one, this is certainly implied. However, the statements and concepts of this paragraph are suitable only in two - dimensional and three - dimensional space. Since the Dirichlet - Voronoi region remains indefinite in space of higher dimension. It will be determined only when the Euler – Poincare equation is fulfilled for it. But this proof in the theory of Voronoi in the case of space of higher dimension is not.

Further, in the Delaunay theory, groups of motion of partitions of *n* - dimensional spaces are considered, i.e. symmetry transformations of these partitions are considered, following the Fedorov theory when considering partitions of two - and three - dimensional spaces. However, symmetry transformations of partitions of spaces of higher dimension cannot be considered if symmetry transformations of the constituent figures of higher dimension are unknown. And they, of course, are unknown, since the figures themselves remained unknown. The symmetry transformations of higher - dimensional polytopes were first consecutively considered in the author's monograph (Zhizhin, 2019 a).

Further in the work of Shtogrin follows the definition of a normal partition of an *n* - dimensional space. "A partition is called normal if for any bodies V_1 and V_2 of this partition in the group *G* of motions that combine this partition, there is at least one movement that takes the body V_1 to the body V_2. The bodies of the normal partition are called stereohedrons. In particular, if *G* is a group of parallel translations, then the partition bodies are called parallelohedra. "

In these definitions, it remained unknown what figures of higher dimension should be, that is, what should be their structure in order for these definitions to be fulfilled. Thus, these definitions do not define the essence of these figures. In this regard, for example, despite these definitions and proofs of a number of theorems (Delaunay, 1961), neither Delaunay himself nor his followers were able to give at least one specific example of a stereohedron of higher dimension.

Of great importance in Delaunay geometry is given to the set of points, which is called the "Delaunay system". This concept was introduced in 1924 and published in 1937 (Delaunay, 1937). Following the work of Galiulin (2003), the Delaunay system is the set of points that satisfies the following two axioms:

1. **Discreteness:** The distance from any point of the set to the nearest point of the same set is greater than or equal to some fixed segment of length *r*;
2. **Coverage:** The distance from any point in space to the point of the system nearest to it is less than or equal to some fixed segment of length *R*.

In the study of the structure of substances in recent decades, phenomena were discovered that were unknown at the time of David Hilbert. These include spontaneous zooming (scaling processes), for example, during second-order phase transitions (Landau, 1937; Kadanoff, 1966; Wilson, 1971); discovery of crystals with the absence of translational symmetry in three-dimensional space (Shehtman, et al., 1984); detection of the fractal nature of matter (Mandelbrott, 1982); hierarchical filling of space with polytopes of higher dimension (Zhizhin, 2012); the highest dimension of most molecules of chemical compounds (Zhizhin, 2018). All these phenomena cannot be described by Delaunay point systems. Zhizhin (2018) introduced the concept of a discrete system of points in which the distance between points asymptotically tends to zero, without reaching exact equality. However, there is no need to introduce the concept of an "empty ball" introduced in Delaunay systems. The possibility of asymptotically reducing the distance between points of a discrete system of points corresponds to the distribution of points in the diffraction patterns of quasicrystals and to the scaling process, i.e. continuous zooming in on a system discovered in recent decades. In connection with the above, it should be considered that the decisive value attributed to systems of Delaunay for describing a discrete world (Galiulin, 2003) is significantly exaggerated and, moreover, its use may lead to incorrect results. For example, in the work of Ryzhkov and Shushbaev (1981), based on the understanding of Delaunay systems, it is argued that with the help of a 4 – cross - polytope one can obtain a correct partition of four - dimensional space. Let us prove that this is not so. We use the representation of 4 - cross - polytopes in the form proposed by Stringham (Stringham, 1880) Figure 1.

Figure 1. 4 - cross - polytope

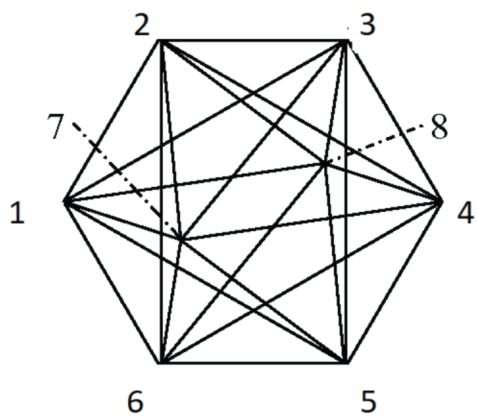
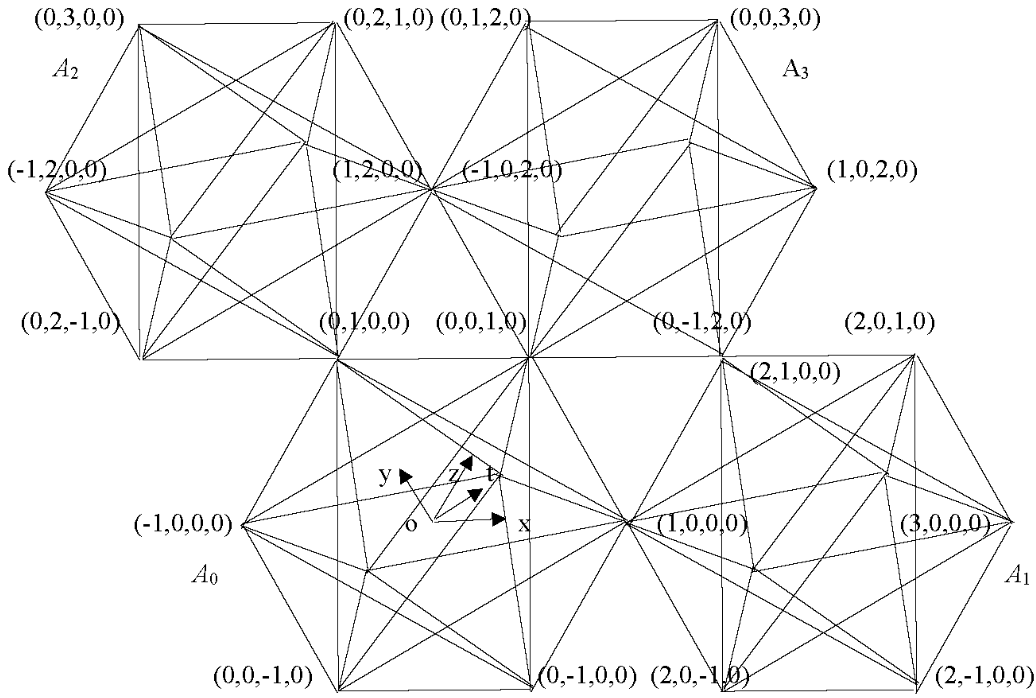


Figure 1 shows that it has 8 vertices, 24 edges, 32 flat triangular faces and 16 tetrahedra as a facet. A feature of cross - polytopes of any dimension is that pairs of oppositely located vertices are not connected by an edge, and there are as many such pairs as the dimension of cross -polytopes. In the case of 4 - cross - polytopes in Figure 1, these are pairs of vertices 1 - 4, 2 - 5, 3 - 6, 7 - 8. Let us introduce the origin of the coordinate system of four - dimensional space (x, y, z, t) in the center of some initial 4 – cross - polytopes A_0 (Figure 2).

The coordinates will be directed in the directions between the vertices that are not connected by edges. Let the distance from the origin to any vertex be equal to unity. Then the coordinates of the vertices indicated in Figure 1 are equal 1(-1, 0, 0, 0), 2(0, 1, 0, 0), 3(0, 0, 1, 0), 4(1, 0, 0, 0), 5 (0, - 1, 0, 0), 6(0, 0,- 1, 0), 7(0, 0, 0, -1), 8(0, 0, 0, -1). We will broadcast polytope A_0 in coordinates x, y, z by two units. Then we get three more 4 - cross - polytopes. The coordinate values of the vertices of all four cross - polytopes are determined by the equalities:

Figure 2. The translation of 4 – cross – polytope at coordinates x, y, z on one step of length 2



$$
\begin{aligned}
&A_0 = \{(-1,0,0,0),(0,1,0,0),(0,0,1,0),(1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1),(0,0,0,1)\},\\
&A_1 = A_0(x+2) = \{(1,0,0,0),(2,1,0,0),(2,0,1,0),(3,0,0,0),(2,-1,0,0),(2,0,-1,0),(2,0,0,-1),\\
&(2,0,0,1)\},\\
&A_2 = A_0(y+2) = \{(-1,2,0,0),(0,3,0,0),(0,0,1,0),(1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1),\\
&(0,0,0,1)\},\\
&A_3 = A_0(z+2) = \{(-1,0,0,0),(0,1,0,0),(0,0,1,0),(1,0,0,0),(0,-1,0,0),(0,0,-1,0),(0,0,0,-1),\\
&(0,0,0,1)\}
\end{aligned}
\qquad (1)
$$

Even without translation along the coordinate *t*, it follows from equalities (1) and Figure 2 that a 4 – cross - polytope cannot fill a four - dimensional space face – to - face, as polytopes A_1 and A_3, as well as polytopes A_3 and A_2, result from the translation matching vertices with different coordinate values. And besides, between polytopes A_0, A_2, A_3, as well as between polytopes A_0, A_1, A_3, areas arise in which a 4 – cross - polytope cannot fit in any way. Q.E.D.

ANALYSIS OF FOUR-DIMENSIONAL DELAUNAY PARALLELOHEDRONS

In 1929, Delaunay gave a classification of parallelohedrons, which he believed had a dimension of 4. This classification includes 51 figures. The principle of constructing four - dimensional parallelohedrons, which Delaunay used, contradicts the existence of polytopes of higher dimension. In particular, in polytopes of higher dimensionality, three - dimensional faces when depicting polytopes in a projection onto a two-dimensional plane, relying on some two -dimensional face, pass through each other (Stringham, 1880; Coxeter, 1963; Grünboum, 1967; Zhizhin, 2019 a). In

Delaunay constructions, three - dimensional faces in projections onto the two - dimensional plane of four - dimensional figures are located on opposite sides of the two -dimensional face on which they rely. This applies to all figures in the Delaunay classification. For none of the figures in the Delaunay classification is there evidence that the figure has dimension 4. A necessary condition for the existence of a multidimensional convex figure is the fulfillment of the Euler - Poincare (Poincare, 1895) equation for this figure. Delaunay does not even mention this condition. It can be shown that for any of the Delaunay classification figures this condition is not fulfilled; therefore, the figures given by Delaunay are not four - dimensional parallelohedrons. We give this proof for three arbitrarily taken figures from the Delaunay classification.

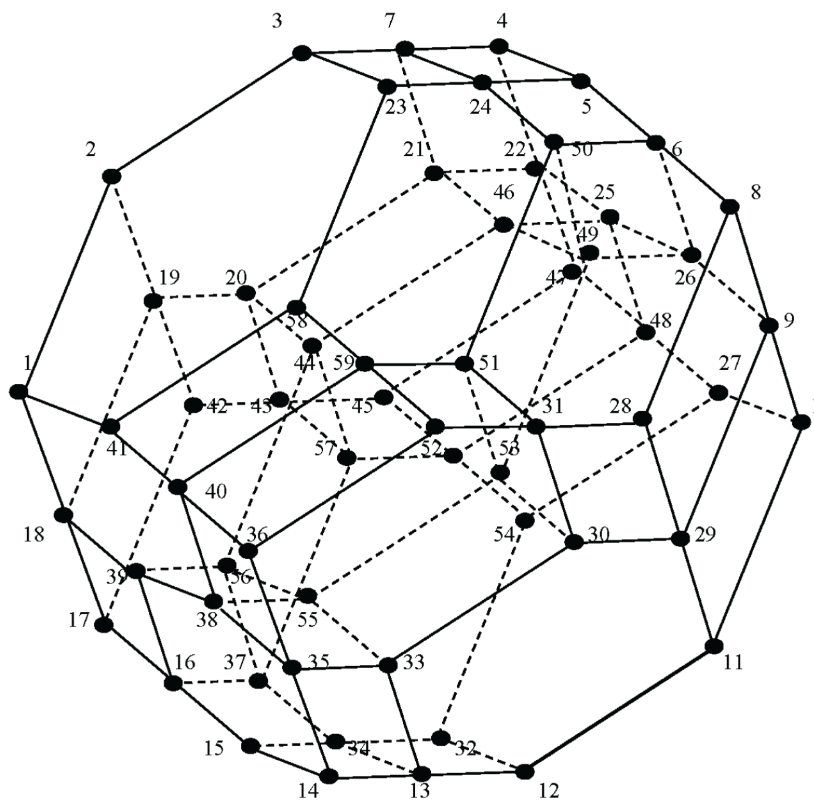
The dimension of a figure is determined by the Euler - Poincare equation (Poincare, 1895):

$$\sum_{i=0}^{d-1} (-1)^i f_i(P) = 1 + (-1)^{d-1} \tag{2}$$

In (2) $f_i(P)$ is the number of faces with dimension *i* in polytope *P* with dimension *d*.

The first figure in the Delaunay classification of four - dimensional parallelohedra is the figure (Delaunay, 1929), composed of 10 three - dimensional figures with 14 flat faces and 20 three-dimensional figures with 8 flat faces (Figure 3).

Figure 3. Parallelohedron 1 from the Delaunay classification



As follows from Figure 3, it contains 60 vertices ($f_0(P_1) = 60$). The edges in this figure are the following segments:

1 – 2, 1 - 18, 1 - 41, 2 - 3, 2 - 19, 3 - 7, 3 - 23, 4 - 7, 4 - 5, 4 - 22, 5 - 24, 5 -6, 6 - 50, 6 - 26, 6 - 8, 7 - 21, 7 - 24, 8 - 9, 8 - 28, 9 - 26, 9 - 10, 9 - 29, 10 - 27, 10 - 11, 11 - 29, 11 - 12, 12 - 13, 12 - 32, 13 - 33, 13 - 34, 13 - 14, 14 - 35, 14 - 15, 15 - 34, 15 - 16, 16 - 37, 16 -39, 16 - 17, 17 - 42, 17 - 18, 18 - 19, 18 - 39, 19 - 20, 19 - 42, 20 - 43, 20 - 44, 20 - 21, 21 - 46, 21 - 22, 22 - 25, 22 - 47, 23 - 58, 23 - 24, 24 - 50, 25 - 46, 25 - 48, 25 - 26, 26 - 49, 27 - 48, 27 -54, 28 - 31, 28 - 29, 29 - 30, 30 - 33, 30 - 53, 30 - 31, 31 - 51, 31 - 60, 32 - 34, 32 - 54, 33 - 35, 33 - 55, 34 - 37, 35 - 38, 35 - 36, 36 - 40, 36 - 60, 37 - 56, 37 - 57, 38 - 39, 38 - 40, 38 - 55, 39 -56, 40 - 59, 40 - 41, 41 - 58, 42 - 43, 43 - 45, 43 - 57, 44 - 46, 44 - 57, 44 - 56, 45 - 52, 46 - 49, 47 - 45, 47 - 48, 48 - 52, 49 - 53, 49 - 50, 50 - 51, 51 - 59, 51 - 53, 52 - 57, 52 - 54, 53 - 55, 55 -56, 58 - 59, 59 - 60

Thus, $f_1(P_1) = 107$.

The listed edges form the following two - dimensional faces:

1 – 2 – 3 – 23 – 58 - 41, 1 – 41 – 40 – 38 – 39 - 18,1 – 2 – 19 - 18, 2 – 3 – 7 – 21 – 20 - 19, 3 – 7 – 24 - 23, 4 – 7 – 21 - 22, 4 – 7 – 5 -24, 4 – 5 – 6 – 26 – 25 - 22, 5 – 6 – 50 - 24, 6 – 8 – 9 - 26, 7 – 24 – 21 – 50 – 46 - 49, 6 – 8 – 28 -31 – 51 - 50, 8 – 9 – 28 - 29, 9 – 10 – 11 - 29, 9 – 26 – 49 – 53 – 30 - 29, 9 – 10 – 27 – 48 – 25 -26, 10 – 27 – 54 – 32 – 12 - 11, 11 – 29 – 30 – 33 – 13 - 12, 12 – 13 – 34 - 32, 13 – 14 – 15 - 34, 13 – 14 – 33 - 35, 14 – 15 – 35 – 38 – 39 - 16, 15 – 34 – 37 - 16, 16 – 37 – 56 - 39, 16 – 17 – 39 -18, 16 – 37 – 57 – 43 – 42 - 17, 17 – 18 – 19 - 42, 18 – 19 – 20 – 44 – 56 - 39, 19 – 20 – 42 - 43, 20 – 21 – 46 - 44, 20 – 21 – 22 – 47 – 45 - 43, 20 – 43 – 44 - 57, 21 – 22 – 25 - 46, 22 – 25 – 48 -47, 23 – 24 – 50 – 51 – 59 - 58, 25 – 26 – 46 - 49, 25 – 46 – 44 – 57 – 52 - 48, 27 – 48 – 52 - 54, 28 – 29 – 30 - 31, 30 – 31 – 60 – 36 – 35 - 33, 30 – 31 – 51 - 53, 31 – 51 – 59 - 60, 32 – 54 – 52 -57 – 37 - 34, 35 – 37 – 38 - 40, 36 – 40 – 59 - 60, 37 – 56 – 44 - 57, 38 – 39 – 55 - 56, 40 – 42 -58 - 59, 44 – 46 – 49 – 53 – 55 - 56, 45 – 47 – 48 - 52

Thus, $f_2(P_1) = 50$.

It should be noted that when calculating the number of edges, it is assumed that the edge does not contain a vertex inside. Otherwise, we are dealing with two edges, i.e. vertices can only be on border of edges. The same situation applies to flat faces. A two - dimensional region is considered a face only when it is not separated by any edge (Zhizhin, 2019 a).

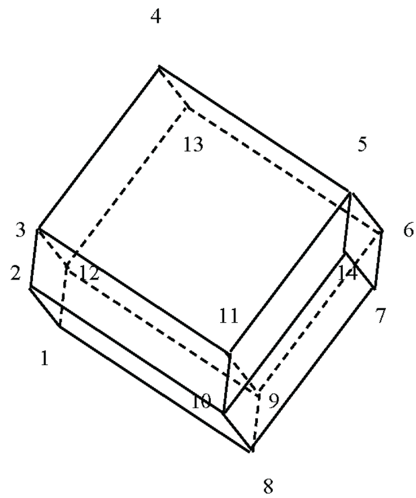
The number of three - dimensional figures included in P_1 according to Delaunay's order is 30. To them we must add the figure formed by the outer surface of P_1 (Zhizhin, 2019 a). Consequently, $f_3(P_1) = 31.$ Substituting the obtained values f_i in the left side of the Euler - Poincare Equation (2) we find:

60 – 107 + 50 – 31 = - 28

The obtained value of the left side of Equation (2) cannot be equal to the right side of Equation (2) for any value of dimension d (the right side can be equal to only 2 or 0). Therefore, the figure indicated by the Delaunay parallelohedron 1 is not a four - dimensional parallelohedron. Moreover, since the Euler - Poincare equation is not satisfied for it, it cannot be considered a convex figure at all.

Consider some simpler figure from the Delaunay classification, for example, figure number 15 (Figure 4).

Figure 4. Parallelohedron 15 from Delaunay classification



From Figure 4 it follows that it has 14 vertices ($f_0(P_{15}) = 14$). The edges in this figure are the following segments:

1 - 12, 1 - 8, 2 - 3, 2 - 10, 3 - 12, 3 - 4, 3 - 11, 4 - 13, 4 - 5, 5 - 6, 5 - 14, 5 -11, 6 - 13, 6 - 9, 6 - 7, 7 - 14, 7 - 8, 8 - 9, 8 - 10, 9 - 12, 9 - 11, 10 - 14, 10 - 11, 12 - 13

Thus, $f_1(P_{15}) = 25$.
The listed edges form the following two - dimensional faces:

1 – 2 – 3 - 12, 1 – 2 – 10 - 8, 1 – 12 – 9 - 8, 2 – 3 – 11 - 10, 3 – 11 – 4 - 5, 3 – 12 – 13 - 4, 4 – 5 – 6 - 13, 5 – 6 – 7 - 14, 5 – 11 – 10 -14, 6 – 9 – 12 - 13, 6 – 7 – 8 - 9, 7 – 8 – 10 - 14, 8 – 9 – 10 – 11

Thus, $f_2(P_{15}) = 13$.
From Figure 4, it follows that it has three - dimensional faces:

1 – 2 – 3 – 12 – 9 – 8 – 10 - 11, 8 – 10 – 11 – 9 – 5 – 6 – 7 - 14, 3 – 12 – 9 – 11 – 4 – 13 – 5 – 6

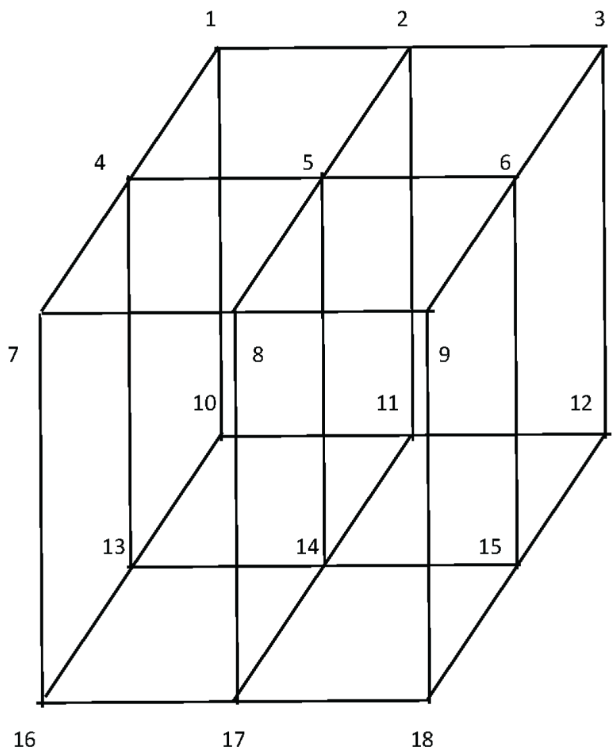
Besides, the three - dimensional face form outer surface of figure. Thus, $f_3(P_{15}) = 4$.
Substituting the obtained values f_i in the left side of the Euler - Poincare Equation (2) we find:

14 – 25 + 13 – 4 = - 2

The obtained value of the left side of Equation (2) cannot be equal to the right side of Equation (2) for any value of dimension d (the right side can be equal to only 2 or 0). Therefore, the figure indicated by the Delaunay parallelohedron 15 is not a four - dimensional parallelohedron. Moreover, since the Euler - Poincare equation is not satisfied for it, it cannot be considered a convex figure at all. When calculating the number of three - dimensional polyhedrons included in a parallelohedron, Delaunay considered three - dimensional figures formed from the combination of several three - dimensional figures as independent three - dimensional figures, in addition to the connected figures. This, firstly, contradicts the accepted condition for counting the number of figures of various dimensions, and secondly, it only further deviates the value of the left side of Equation (2) from the value of the right side of this equation.

The simplest figure in the Delaunay classification is Figure 5.

Figure 5. Parallelohedron 13 from Delaunay classification



The flaws of this classification are most clearly identified in this figure.

In fact, it is a folded four dice. The number of vertices in this figure is 18 ($f_0(P_{13}) = 18$). The edges in this figure are the following segments:

1 - 2, 1 - 4, 1 - 10, 2 - 3, 2 - 11, 2 - 5, 3 - 12, 3 - 6, 4 - 13, 4 - 7, 4 - 5, 5 - 6, 5 - 14, 5 - 8, 6 - 15, 6 - 9, 16 - 7, 7 - 8, 8 - 9, 8 - 17, 9 - 18, 10 - 11, 10 - 13, 11 - 12, 14 - 11, 12 - 15, 13 - 14, 13 - 16, 14 - 15, 14 - 17, 15 - 18, 16 - 17, 17 – 18

Thus, $f_1(P_{13}) = 33$.

The listed edges form the following two-dimensional faces:

1 – 2 – 4 - 5, 1 – 2 – 10 - 11, 1 – 4 -10 - 13, 2 – 3 – 11 - 12, 2 – 3 – 5 - 6, 2 – 11 – 14 - 5, 3 – 12 – 6 - 15, 4 – 5 – 13 - 14, 4 – 5 – 7 - 8, 4 – 7 – 13 - 16, 5 – 6 – 14 - 15, 5 – 6 – 9 - 8, 5 – 8 – 14 - 17, 6 – 9 – 15 - 18, 7 – 8 – 16 - 17, 8 – 9 – 17 – 18

Thus, $f_2(P_{13}) = 16$.

From Figure 5, it follows that it has four three- dimensional faces:

1 – 2 – 4 – 5 – 10 – 11 – 13 -14, 2 – 3 – 5 – 6 – 11 – 12 – 14 - 15, 4 – 5 – 7 – 8 – 13 – 14 – 16 - 17, 5 – 6 – 8 – 9 – 14 – 15 -17 – 18

Besides, the three - dimensional face form outer surface of figure. Thus, $f_3(P_{13}) = 5$.

Substituting the obtained values f_i in the left side of the Euler - Poincare Equation (2) we find:

18 – 33 + 16 – 5 = - 4

The obtained value of the left side of Equation (2) cannot be equal to the right side of equation (2) for any value of dimension d (the right side can be equal to only 2 or 0). Therefore, the figure indicated by the Delaunay parallelohedron 13 is not a four - dimensional parallelohedron.

The noted disadvantages apply to all Delaunay classification figures. Therefore, all these figures are not four - dimensional parallelohedrons. And this is natural. Roughly speaking, it is impossible to change the dimension of space by folding cubes.

Subsequently, Delaunay attempted to construct the theory of stereohedrons as part of parallelohedrons of higher dimension (Delaunay, 1961; Delaunay, Sandakova, 1961), using the definitions of stereohedrons by Fedorov (1885). However, when this was used, as was later discovered (Zhizhin, 2019 a), erroneous assumptions and not a single concrete image of a stereohedron of dimension higher than three were obtained.

We now verify the possibility of fulfilling the Euler - Poincaré equation for the figure obtained by Shtogrin (1973), which he considered as 52 four - dimensional parallelohedron, i.e. additional to the Delaunay classification. This figure is shown in Figure 6.

As follows from Figure 6, it contains 21 vertices ($f_0(P_{52}) = 21$). The edges in this figure are the following segments:

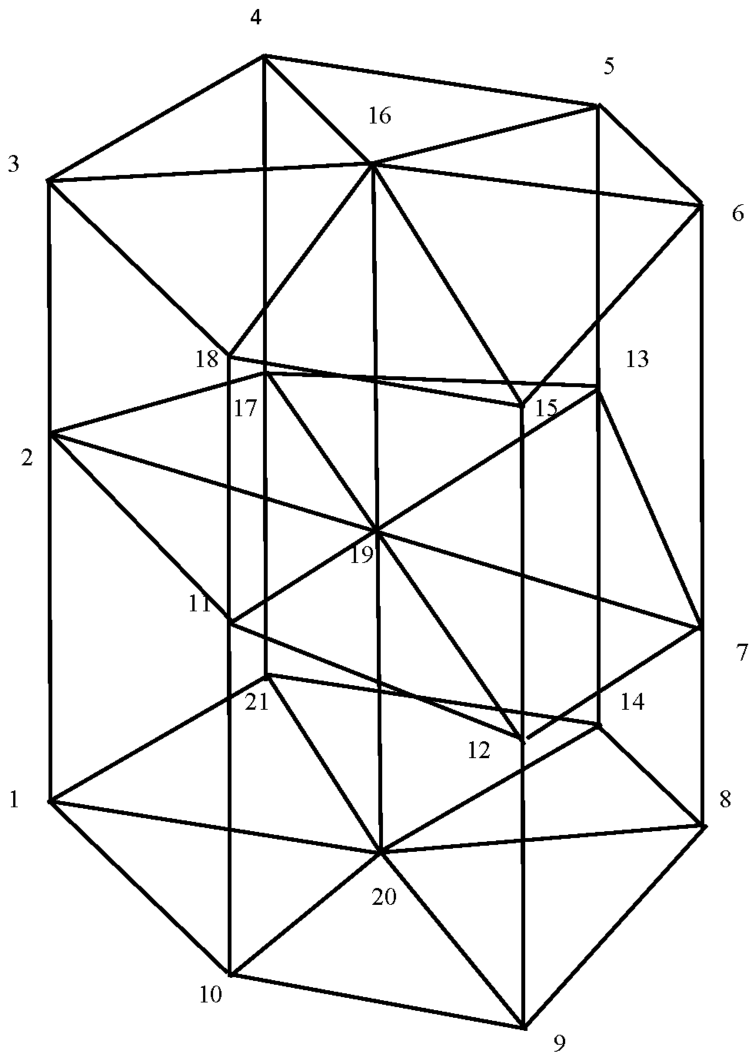
1 – 10, 1 - 21, 1 - 41, 1 - 2, 2 - 17, 3 - 2, 2 - 19, 2 - 11, 4 - 3, 3 - 16, 3 - 18, 5 - 4, 4 - 16, 4 - 17, 5 - 16, 5 – 6, 5 - 13, 6 - 16, 6 - 15, 6 - 7, 7 - 13, 7 - 19, 7 - 21, 7 - 8, 8 - 14, 8 - 20, 8 - 9, 9 - 12, 9 - 20, 9 - 10, 10 - 20, 10 – 11, 11 - 18, 11 - 12, 11 - 19, 12 - 15, 12 - 19, 13 - 19, 13 - 17, 13 - 14, 14 - 20, 14 - 21, 15 - 16, 15 - 18, 16 - 18, 16 - 19, 17 - 19, 17 - 21, 19 - 20, 20 – 21

Thus, $f_1(P_{52}) = 50$.

The listed edges form the following two - dimensional faces:

1 – 2 – 11 – 10, 1 – 21 – 20, 1 – 20 – 10, 2 – 1 – 17 – 21, 3 – 2 – 16 - 19, 2– 3 – 18 - 11, 2 – 17 – 19, 2 – 19 – 11, 3 – 4 – 16, 3 – 16 – 18, 4 – 5 – 13 – 17, 4 – 16 – 17 - 19, 4 – 5 – 16, 5 – 6 – 16, 5 – 6 – 13 – 7, 5 – 16 – 13 – 19, 6 – 7 – 16 – 19, 6 – 15 – 16, 6 – 7 – 12 - 15, 7 – 13 – 19, 7 – 19 – 20 - 8, 7 – 8 – 12 – 9, 7 – 8 – 13 - 14, 8 – 20 – 9, 8 – 14 – 20, 9 – 12 – 19 – 20, 9 – 20 – 10, 9 – 10 – 11 – 12, 10– 11 – 19 - 20, 11 – 18 – 16 - 19, 11 – 19 – 12, 11 – 18 – 15 - 12, 12 – 15 – 16 - 19, 13 – 14 – 19 - 20, 13 – 17 – 19, 14 – 20 – 21, 15 – 16 – 18, 17 – 19 – 20 – 21

Figure 6. Four - dimensional Shtogrin parallelohedron



Thus, $f_2(P_{52}) = 39$.

It should be noted that when calculating the number of edges, it is assumed that the edge does not contain a vertex inside. Otherwise, we are dealing with two edges, i.e. vertices can only be on border of edges. The same situation applies to flat faces. A two - dimensional region is considered a face only when it is not separated by any edge (Zhizhin, 2019 a).

The number of three - dimensional figures included in P_{52} is 13:

1 – 20 – 10 – 2 – 19 – 11, 1 – 21 – 20 – 2 – 17 – 19, 2 – 11 – 19 – 3 – 16 – 18, 2 – 17 – 19 – 3 – 4 – 16, 4 – 5 – 16 – 13 - 17 – 19, 5 – 16 – 6 – 13 – 7 – 19, 6 – 7 – 16 – 15 – 12 – 19, 7 – 13 – 19 – 8 – 14 – 20, 7 – 8 – 19 – 20 – 12 – 9, 9 – 12 – 19 – 20 – 10 – 11, 11 – 12 – 1 9 – 15 – 16 – 18, 13 – 14 – 19 – 20 – 17 – 21, 1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10 – 11 – 12 – 13 – 14 – 15 – 16 – 17 – 18 – 19 – 20 – 21

Consequently, $f_3(P_{52}) = 13$. Substituting the obtained values f_i in the left side of the Euler - Poincare Equation (2) we find:

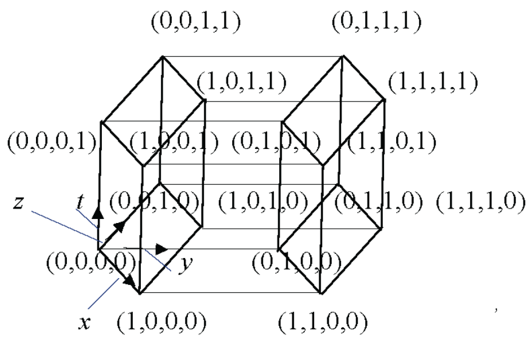
21– 50 + 39 – 13 = - 3

The obtained value of the left side of Equation (2) cannot be equal to the right side of equation (2) for any value of dimension d (the right side can be equal to only 2 or 0). Therefore, the figure indicated by the parallelohedron 52 is not a four - dimensional parallelohedron. Moreover, since the Euler - Poincare equation is not satisfied for it, it cannot be considered a convex figure at all.

CUBE AS A POLYTOPIC PRISMAHEDRON

Theorem 1: *4* – cube is 4 – dimensional parallelohedron in 4 – dimensional space.
Proof: A four - dimensional cube is the simplest version of a polytopic prismahedron, since it is the product of a three - dimensional cube and a one - dimensional segment (Figure 7). Can to

Figure 7. 4 - cube



introduce one of the vertices of the 4 – cube origin of the four - dimensional space (*x, y, z, t*). Orient the coordinates, such as indicated in Figure 7.

Assume that the length of each edge is equal to 1. Then, each vertex of 4 - cube can be associated with a set of integers (Figure 7). Then:

$$A_0 = [(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,0,0),(0,1,1,0),(0,1,0,1),(0,0,1,1),\\ (1,1,1,0),(1,1,0,1),(1,0,1,1),(0,1,1,1),(1,1,1,1)], \tag{3}$$

$$A_1 = A_0(x+1) = [(1,0,0,0),(2,0,0,0),(1,1,0,0),(1,0,1,0),(1,0,0,1),(2,1,0,0),(2,0,1,0),\\ (2,0,0,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),(2,1,1,0),(2,0,1,1),(1,1,1,1),(2,1,1,1)],$$

$$A_2 = A_0(z+1) = [(0,0,1,0),(1,0,1,0),(0,1,1,0),(0,0,2,0),(0,0,1,1),(1,1,1,0),(1,0,2,0),(1,0,1,1),\\ (0,1,2,0),(0,1,1,1),(0,0,2,1),(1,1,2,0),(1,1,1,1),(1,0,2,1),(0,1,2,1),(1,1,2,1)],$$

$$A_3 = A_0(x+1,z+1) = A_1(z+1) =$$
$$[(1,0,1,0),(2,0,1,0),(1,1,1,0),(1,0,2,0),(1,0,1,1,),(2,1,1,0),(2,0,2,0),(2,0,1,1),$$
$$(1,1,2,0),(1,1,1,1),(1,0,2,1),(2,1,2,0),(2,1,1,1),(2,0,2,1),(1,1,2,1),(2,1,2,1)],$$

$$A_4 = A_0(y+1,z+1) = A_2(y+1) =$$
$$[(0,1,1,0),(1,1,1,0),(0,2,1,0),(0,1,2,0),(0,1,1,1),(1,2,1,0),(1,1,2,0),(1,1,1,1),$$
$$(0,2,2,0),(0,2,1,1),(0,1,2,1),(1,2,2,0),(1,2,1,1),(0,2,2,1),(1,2,2,1)],$$

$$A_5 = A_0(x+1,y+1,z+1) = A_3(y+1) =$$
$$[(1,1,1,0),(2,1,1,0),(1,2,1,0),(1,1,2,0),(1,1,1,1),(2,2,1,0),(2,1,2,0),(2,1,1,1),$$
$$(1,2,2,0),((1,2,1,1),(1,1,2,1),(2,2,2,0),(2,2,1,1),(2,1,2,1),(1,2,2,1),(2,2,2,1)],$$

$$A_6 = A_0(y+1) = [(0,1,0,0),(1,1,0,0),(0,2,0,0),(0,1,1,0),(0,1,0,1),(1,2,0,0),(0,2,1,0),(0,2,0,1)],$$

$$A_7 = A_0(y+1,x+1) = A_1(y+1) =$$
$$[(1,1,0,0),(2,1,0,0),(1,2,0,0),(1,1,1,0),(1,1,0,1),(2,2,0,0),(2,1,1,0),(2,1,0,1),$$
$$(1,2,1,0),(1,2,0,1),(1,1,1,1),(2,2,1,0),(2,2,0,1),(2,1,1,1),(1,2,1,1),(2,2,1,1)],$$

$$A_8 = A_0(t+1) =$$
$$[(0,0,0,1),(1,0,0,1),(0,1,0,1),(0,0,1,1),(0,0,0,2),(1,1,0,1),(1,0,1,1),$$
$$(1,0,0,2),(0,1,1,1),(0,1,0,2),(0,0,1,2),(1,1,1,1),(1,1,0,2),(1,0,1,2),(0,1,1,2),(1,1,1,2)],$$

$$A_9 = A_0(y+1,t+1) = A_6(t+1) =$$
$$[(0,1,0,1),(1,1,0,1),(0,2,0,1),(0,1,1,1),(0,1,0,2),(1,2,0,1),(1,1,1,1),(1,1,0,2),$$
$$(0,2,1,1),(0,2,0,2),(0,1,1,2),(1,2,1,1),(1,2,0,2),(1,1,1,2),(0,2,1,2),(1,2,1,2)],$$

$$A_{10} = A_0(x+1,y+1,t+1) = A_7(t+1) =$$
$$[(1,1,0,1),(2,1,0,1),(1,2,0,1),(1,1,1,1),(1,1,0,2),(2,2,0,1),(2,1,1,1),(2,1,0,2),(1,2,1,1),$$
$$(1,2,0,2),(1,1,1,2),(2,2,1,1),(2,2,0,2),(2,1,1,2),(1,2,1,2),(2,2,1,2)],$$

$$A_{11} = A_0(x+1,t+1) = A_1(t+1) =$$
$$[(1,0,0,1),(2,0,0,1),(1,1,0,1),(1,0,1,1),(1,0,0,2),(2,1,0,1),(2,0,1,1),(2,0,0,2),$$
$$(1,1,1,1),(1,1,0,2),(1,0,1,2),(2,1,1,1),(2,1,0,2),(2,0,1,2),(1,1,1,2),(2,1,1,2)],$$

$$A_{12} = A_0(y+1,z+1) = A_2(t+1) =$$
$$[(0,0,1,1),(1,0,1,1),(0,1,1,1),(0,0,2,1),(0,0,1,2),(1,1,1,1),(1,0,2,1),(1,0,1,2),$$
$$(0,1,2,1),(0,1,1,2),(0,0,2,2),(1,1,2,1),(1,1,1,2),(1,0,2,2),(0,1,2,2),(1,1,2,2)],$$

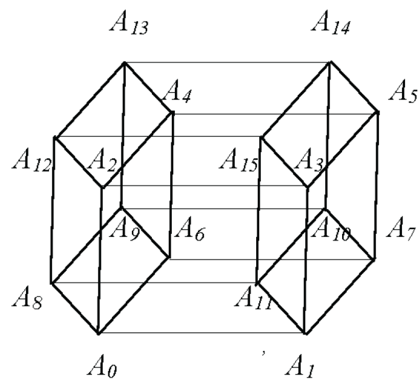
$$A_{13} = A_0(z+1,y+1,t+1) = A_4(t+1) =$$
$$[(0,1,1,1),(1,1,1,1),(0,2,1,1),(0,1,2,1),(0,2,2,1),(0,1,1,2),(1,2,1,1),(0,2,1,2),$$
$$(1,1,2,1),(1,1,1,2),(0,1,2,2),(1,2,2,1),(1,2,1,2),(1,1,2,2),(0,2,2,2),(1,2,2,2)],$$

$$A_{14} = A_0(x+1,y+1,z+1,t+1) = A_5(t+1) =$$
$$[(1,1,1,1),(2,1,1,1),(1,2,1,1),(1,1,2,1),(1,1,1,2),(2,2,1,1),(2,1,2,1),(2,1,1,2),$$
$$(1,2,2,1),(1,2,1,2),(1,1,2,2),(2,2,2,1),(2,2,1,2),(2,1,2,2),(1,2,2,2),(2,2,2,2)],$$

$$A_{15} = A_0(x+1, t+1, z+1) = A_3(t+1) =$$
$$[(1,0,1,1),(2,0,1,1),(1,1,1,1),(1,0,2,1),(1,0,1,2),(2,1,1,1),(2,0,2,1),(2,0,1,2),$$
$$(1,1,2,1),(1,1,1,2),(1,0,2,2),(2,1,2,1),(2,1,1,2),(2,0,2,2),(1,1,2,2),(2,1,2,2)].$$

Representing the 4 - cubes $A_0 \div A_{15}$ dots in three - dimensional space, can get the 4 - cube again (4 - cube A). Moreover, the edges of the 4 - cube A correspond to possible changes in the values of one of the coordinates of the vertices of the 4 - cubes unit (Figure 8).

Figure 8. The 4 - cube A from 16 4 - cubes



In addition, each edge of the 4 - cube A in Figure 8 can be considered as an element of the overall two 4 - cubes, connected by an edge. Using the coordinate expression 4 - cubes (3) can be analytically determined. In Table 1 geometry elements are common to each pair of the 4 - cubes connected by an edge in Figure 8 are listed.

From Table 1 it follows that for two 4 - cubes shifted parallel to each other in the same coordinate by the edge length, a 3 - cube is common, i.e. facet. However, shifting 3 - cubes by the length of an edge over a larger number of coordinates leads to a change in the common element of the cubes. Thus, the shift of 3 - cubes by the length of the edge in two coordinates corresponds to the diagonals of flat faces of the 4 - cube A. Using (3), it can be shown that in this case squares are common elements of 4 - cubes. For example, diagonals A_8A_{13}, A_9A_{12} correspond to a common square element (0,1,1,1)(1,1,1,1)(0,1,1,2)(1,1,1,2); diagonals A_0A_4, A_2A_6 correspond to a common square element (0,1,1,0)(1,1,1,0)(0,1,1,1)(1,1,1,1); diagonals A_2A_8, A_0A_{12} correspond to a common square element (0,0,1,1)(1,0,1,1)(0,0,1,1)(1,1,1,1); and go on. Similarly, it can be shown that a shift of 4 - cubes by the length of an edge along three coordinates corresponds to a common element - an edge. When 4 - squares are shifted in four coordinates by the edge length, the common element of 4 - cubes is a vertex. A shift along any coordinate by two edge lengths (or more) leads to the absence of common elements in 4 - cubes.

Since in the vicinity of any 4 - cube at a distance of the length of the edge along each coordinate there is another 4 - cube having a common facet with a central 4 - cube, and on the diagonal of two coordinates there is a 4 - cube having a common flat face with the central 4 - cube, then the 4 - cube during translation fills the entire four - dimensional space without gaps. This means that the 4 - cube is a four - dimensional parallelohedron.

Table 1. Common elements of the 4 - cubes

Edge of the 4-Cube A	Common Cube of the 4-Cubes	Edge of the 4-Cube A	Common Cube of the 4-Cubes
$A_0 A_1$	(1,0,0,0)(1,1,0,0)(1,0,1,0)(1,0,0,1) (1,1,1,0)(1,1,0,1)(1,0,1,1)(1,1,1,1)	$A_5 A_{14}$	(1,1,1,1)(1,2,1,1)(2,1,1,1)(2,2,1,1) (2,1,2,1)(1,2,2,1)(2,2,2,1)(1,1,2,1)
$A_0 A_2$	(0,0,1,0)(1,0,1,0)(0,1,1,0)(0,0,1,1) (1,1,1,0)(1,0,1,1)(0,1,1,1)(1,1,1,1)	$A_6 A_7$	(1,1,1,0)(1,1,0,0)(1,2,0,0)(1,1,0,1) (1,2,1,0)(1,2,0,1)(1,1,1,1)(1,2,1,1)
$A_0 A_6$	(0,1,0,0)(1,1,0,0)(0,1,1,0)(0,1,0,1) (1,1,1,0)(1,1,0,1)(0,1,1,1)(1,1,1,1)	$A_6 A_9$	(0,1,0,1)(0,2,0,1)(1,1,0,1)(0,1,1,1) (1,1,1,1)(0,2,1,1)(1,2,1,1)(0,2,0,1)
$A_0 A_8$	(0,0,0,1)(0,1,0,1)(0,0,1,1)(1,1,0,1) (1,0,1,1)(0,1,1,1)(1,1,1,1)(1,0,0,1)	A_7A_{10}	(1,1,0,1)(1,2,0,1)(1,1,1,1)(2,2,0,1) (1,2,1,1)(2,2,1,1)(2,1,0,1)(2,1,1,1)
A_1A_3	(1,0,1,0)(2,0,1,0)(1,1,1,0)(1,0,1,1) (2,1,1,0)(2,0,1,1)(1,1,1,1)(2,1,1,1)	$A_8 A_{11}$	(1,0,0,1)(1,1,0,1)(1,0,1,1)(1,0,0,2) (1,1,1,1)(1,1,0,2)(1,0,1,2)(0,1,1,2)
A_7A_1	(1,1,0,0,)(2,1,0,0)(1,1,1,0)(1,1,0,1) (2,1,1,0)(2,1,0,1)(1,1,1,1)(2,1,1,1)	A_8A_9	(1,1,1,1)(1,1,0,2)(0,1,1,2)(1,1,1,2) (0,1,0,1)(1,1,0,1)(0,1,1,1)(0,1,0,2)
A_1A_{11}	(1,0,0,1)(2,0,0,1)(1,1,0,1)(1,0,1,1) (2,1,0,1)(2,0,1,1)(1,1,1,1)(2,1,1,1)	A_8A_{12}	(0,0,1,1)(0,1,1,1)(1,0,1,1)(0,0,1,2) (1,1,1,1)(1,0,1,2)(0,1,1,2)(1,1,1,2)
A_2A_3	(1,0,1,0)(1,1,1,0)(1,0,1,1)(0,1,2,0) (1,1,2,0)(1,1,1,1)(1,0,2,1)(1,1,2,0)	A_9A_{10}	(1,1,0,1)(1,2,0,1)(1,1,1,1)(1,1,0,2) (1,2,1,1)(1,1,1,2)(1,2,0,2)(1,1,1,2)
A_2A_4	(0,1,1,0)(1,1,1,0)(0,1,2,0)(0,1,1,1) (1,1,2,0)(1,10,1,1)(0,1,2,1)(1,1,2,1)	A_9A_{13}	(1,2,1,1)(0,1,1,1)(1,1,1,1)(0,2,1,1) (0,1,1,2)(1,2,1,2)(1,1,1,2)(0,2,1,2)
A_2A_{12}	(0,0,1,1)(0,1,1,1)(0,0,2,1)(1,1,1,1) (1,1,2,1)(0,1,2,1)(1,0,2,1)(1,0,1,1)	$A_{11}A_{10}$	(1,1,1,2)(1,1,0,1)(1,1,1,1)(1,1,0,2) (2,1,1,1)(2,1,0,2)(2,1,1,2)(2,1,0,1)
A_3A_5	(1,1,1,0)(2,1,1,0)(1,1,1,1)(2,1,1,1) (1,1,2,1)(2,1,2,1)(2,1,2,0)(2,1,1,1)	$A_{14}A_{10}$	(1,1,1,1)(1,2,1,1)(2,1,1,1)(1,1,1,2) (2,2,1,1)(2,1,1,2)(1,2,1,2)(2,2,1,2)
A_3A_{15}	(2,1,1,1)(1,0,1,1)(2,0,1,1)(1,1,1,1) (1,0,2,1)(2,0,2,1)(1,1,2,1)(2,1,2,1)	$A_{11}A_{15}$	(1,0,1,1)(1,1,1,1)(2,0,1,1)(1,0,1,2) (2,1,1,1)(2,0,1,2)(1,1,1,2)(2,1,1,2)
A_4A_5	(1,1,1,0)(1,2,1,0)(1,1,2,0)(1,1,1,1) (1,2,2,0)(1,2,1,1)(1,1,2,1)(1,2,2,0)	$A_{13}A_{12}$	(0,1,1,1)(1,1,1,1)(0,1,2,1)(0,1,1,2) (1,1,2,1)(1,1,1,2)(1,1,2,2)(0,1,2,2)
A_4A_6	(1,1,1,0))(0,2,1,0)(0,1,1,1)(1,2,1,0) (1,1,1,1)(0,2,1,1)(1,2,1,1)(0,1,1,0)	$A_{12}A_{15}$	(1,0,1,1)(1,1,1,1)(1,0,1,2)(1,0,2,1) (1,1,1,2)(1,0,2,2)(1,1,2,2)(1,1,2,1)
A_4A_{13}	(0,1,1,1)(1,1,1,1)(0,2,1,1)(0,1,2,1) (1,2.1,1)(1,1,2,1)(0,2,2,1)(1,2,2,1)	$A_{13}A_{14}$	(1,1,1,2)(1,2,1,2)(1,1,2,2)(1,2,2,2) (1,1,1,1)(1,1,2,1)(1,2,1,1)(1,2,2,1)
A_7A_5	(1,1,1,0)(2,1,1,0)(1,2,1,0)(1,1,1,1) (2,2,1,0)(2,1,1,1)(1,2,1,1)(2,2,1,1)	$A_{14}A_{15}$	(1,1,1,2)(2,1,2,1)(2,1,1,2)(1,1,2,2) (1,1,1,1)(2,1,1,1)(1,1,2,1)(2,1,2,2)

POLYTOPIC PRISMAHEDRON ON BASES OF SIMPLEXES

Partition of Space by the 4 - Quadrangular Prism

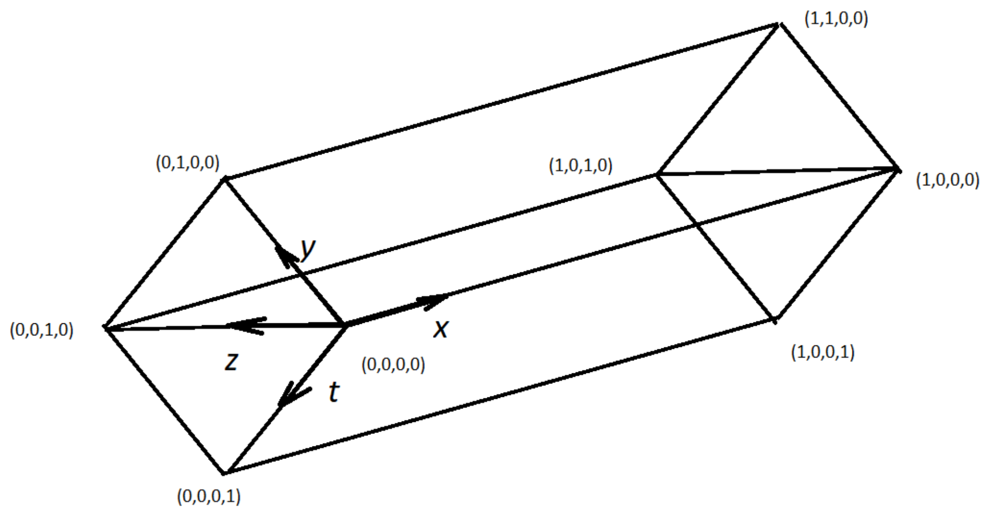
The simplest element of a polytope of simplex type that does not coincide with elements of a polytope of type cube is a triangle. If we multiply the triangle by a one - dimensional segment, then we get a triangular prism. Is it possible to fill the space using a triangular prism, applying them to each other along whole flat faces? Can. If the triangle is equipotential, which can be assumed, then as a result of applying triangular prisms to each other along the side faces, we obtain a quadrangular prism, each of the bases of which are composed of two triangles (Figure 9). Denote this quadrangular prism 4 - quadrangular prism.

It is easy to verify that the dimension of such a construction is 4. Indeed, The number of vertices in this Figure 9 is 8, i.e. $f_0 = 5$; the number of edges is 14, i.e. $f_1 = 14$; the number of planar faces is 9, i.e. $f_2 = 9$; the number of three - dimensional figures is 3 (two triangular prisms and one quadrangular prism), i.e. $f_3 = 3$. To prove the statement, we use the Euler - Poincare Equation (2).

Substituting the obtained values into the Euler - Poincare equation, we obtain:

$$8 - 14 + 9 - 3 = 0$$

Figure 9. 4 - quadrangular prism



We see that Equation (2) is satisfied for $n = 4$. This proves that two triangular prisms folded along the lateral face form a figure with dimension 4. Obviously, this figure is a four - dimensional parallelohedron, since its translation will fill the whole space without gaps. Interestingly, in the case of translation, four - dimensional parallelohedra in this case adjoin each other along two -dimensional faces, either quadrangular or triangular. The neighboring four - dimensional parallelohedra have common two - dimensional faces. This once again proves that the fallacy of the assertion of the Delaunay theory that neighboring four - dimensional parallelohedra when dividing the space must necessarily have common elements of dimension one less than the dimension of parallelohedra, i.e. be three - dimensional. Neighboring four - dimensional parallelohedra can have common triangular faces not during translation, but when they are rotated by 30 degrees relative to each other.

Can to introduce one of the vertices of the 4 - quadrangular prism origin of the four - dimensional space (*x, y, z, t*). Orient the coordinates, such as indicated in Figure 9.

Assume that the length of each edge is equal to 1. Then, each vertex of the 4 - quadrangular prism can be associated with a set of integers (Figure 9). Then:

$$A_0 = (0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (1, 0, 0, 1) \quad (4)$$

$A_1 = A_0(x+1) = (1, 0, 0, 0), (2, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (2, 1, 0, 0),$
$(2, 0, 1, 0), (2, 0, 0, 1),$

$A_2 = A_0(z+1) = (0, 0, 1, 0), (1, 0, 1, 0), (0, 1, 1, 0), (0, 0, 2, 0), (0, 0, 1, 1), (1, 1, 1, 0),$
$(1, 0, 2, 0), (1, 0, 1, 1),$

$A_3 = A_0(x+1, z+1) = A_1(z+1) = (1, 0, 1, 0), (2, 0, 1, 0), (1, 1, 1, 0), (1, 0, 2, 0), (1, 0, 1, 1),$
$(2, 1, 1, 0), (2, 0, 2, 0), (2, 0, 1, 1),$

$A_4 = A_0(y+1, z+1) = A_2(y+1) =$ (0, 1, 1, 0), (1, 1, 1, 0), (0, 2, 1, 0), (0, 1, 2, 0), (0, 1, 1, 1), (1, 2, 1, 0), (1, 1, 2, 0), (1, 1, 1, 1),

$A_5 = A_0(x+1, y+1, z+1) = A_3(y+1) =$ (1, 1, 1, 0), (2, 1, 1, 0), (1, 2, 1, 0), (1, 1, 2, 0), (1, 1, 1, 1), (2, 2, 1, 0), (2, 1, 2, 0), (2, 1, 1, 1),

$A_6 = A_0(y+1) =$ (0, 1, 0, 0), (1, 1, 0, 0), (0, 2, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1), (1, 2, 0, 0), (1, 1, 1, 0), (1, 1, 0, 1),

$A_7 = A_0(y+1, x+1) = A_1(y+1) =$ (1, 1, 0, 0), (2, 1, 0, 0), (1, 2, 0, 0), (1, 1, 1, 0), (1, 1, 0, 1), (2, 2, 0, 0), (2, 1, 1, 0), (2, 1, 0, 1),

$A_8 = A_0(t+1) =$ (0, 0, 0, 1), (1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (0, 0, 0, 2), (1, 1, 0, 1), (1, 0, 1, 1), (1, 0, 0, 2),

$A_9 = A_0(y+1, t+1) = A_6(t+1) =$ (0, 1, 0, 1), (1, 1, 0, 1), (0, 2, 0, 1), (0, 1, 1, 1), (0, 1, 0, 2), (1, 2, 0, 1), (1, 1, 1, 1), (1, 1, 0, 2),

$A_{10} = A_0(x+1, y+1, t+1) = A_7(t+1) =$ (1, 1, 0, 1), (2, 1, 0, 1), (1, 2, 0, 1), (1, 1, 1, 1), (1, 1, 0, 2), (2, 2, 0, 1), (2, 1, 1, 1), (2, 1, 0, 2),

$A_{11} = A_0(x+1, t+1) = A_1(t+1) =$ (1, 0, 0, 1), (2, 0, 0, 1), (1, 1, 0, 1), (1, 0, 1, 1), (1, 0, 0, 2), (2, 1, 0, 1), (2, 0, 1, 1), (2, 0, 0, 2),

$A_{12} = A_0(y+1, z+1) = A_2(t+1) =$ (0, 0, 1, 1), (1, 0, 1, 1), (0, 1, 1, 1), (0, 0, 2, 1), (0, 0, 1, 2), (1, 1, 1, 1), (1, 0, 2, 1), (1, 0, 1, 2),

$A_{13} = A_0(z+1, y+1, t+1) = A_4(t+1) =$ (0, 1, 1, 1), (1, 1, 1, 1), (0, 2, 1, 1), (0, 1, 2, 1), (0, 2, 2, 1), (0, 1, 1, 2), (1, 2, 1, 1), (0, 2, 1, 2),

$A_{14} = A_0(x+1, y+1, z+1, t+1) = A_5(t+1) =$ (1, 1, 1, 1), (2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2), (2, 2, 1, 1), (2, 1, 2, 1), (2, 1, 1, 2),

$A_{15} = A_0(x+1, t+1, z+1) = A_3(t+1) =$ (1, 0, 1, 1), (2, 0, 1, 1), (1, 1, 1, 1), (1, 0, 2, 1), (1, 0, 1, 2), (2, 1, 1, 1), (2, 0, 2, 1), (2, 0, 1, 2)

Representing of 4 - quadrangular prisms $A_0 \div A_{15}$ dots in four - dimensional space, can get the 4 - cube again (4 - cube A1, Figure 8, in which the polytopes at vertices correspondent the equality (4)). Moreover, the edges of the 4 - cube A1 correspond to possible changes in the values of one of the coordinates of the vertices of the 4 - quadrangular prisms unit.

In addition, each edge of the 4 - cube A1 in Figure 10 can be considered as an element of the overall two the 4 - quadrangular prisms, connected by an edge. Using the coordinate expression of the 4 - quadrangular prisms (4) can be analytically determined. In Table 2 geometry elements are common to each pair of the 4 - quadrangular prisms connected by an edge in Figure 10 are listed.

Table 2. Common elements of the 4 - quadrangular prisms

Edge of the 4-Cube A1	Common Elements of the 4 - Quadrangular Prisms	Edge of the 4-Cube A1	Common Cube of the 4 - Quadrangular Prisms
A_0 A_1	(1,0,0,0)(1,1,0,0)(1,0,1,0)(1,0,0,1)	A_5 A_{14}	(1,1,1,1)(2,1,1,1)
A_0 A_2	(0,0,1,0)(1,0,1,0)	A_6 A_7	(1,1,1,0)(1,1,0,0)(1,2,0,0)(1,1,0,1)
A_0 A_6	(0,1,0,0)(1,1,0,0)	A_6 A_9	(0,1,0,1)(1,1,0,1)
A_0 A_8	(0,0,0,1)(1,0,1,0)(1,0,0,1)	A_7A_{10}	(1,1,0,1)(2,1,0,1)
A_1A_3	(1,0,1,0)(2,0,1,0)	A_8 A_{11}	(1,0,0,1)(1,1,0,1)(1,0,1,1)(1,0,0,2)
A_7A_1	(1,1,0,0,)(2,1,0,0)	A_8A_9	(0,1,0,1)(1,1,0,1)
A_1A_{11}	(1,0,0,1)(2,0,0,1)	A_8A_{12}	(0,0,1,1)(1,0,1,1)
A_2A_3	(1,0,1,0)(1,0,2,0)(1,0,1,1)	A_9A_{10}	(1,1,0,1)(1,2,0,1)(1,1,1,1)(1,1,0,2)
A_2A_4	(0,1,1,0)(1,1,1,0)	A_9A_{13}	(0,1,1,1)(1,1,1,1)
A_2A_{12}	(0,0,1,1)(1,0,1,1)	$A_{11}A_{10}$	(1,1,0,1)(2,1,0,1)
A_3A_5	(1,1,1,0)(2,1,1,0)	$A_{14}A_{10}$	(1,1,1,1)(2,1,1,1)
A_3A_{15}	(1,0,1,1)(2,0,1,1)	$A_{11}A_{15}$	(1,0,1,1)(2,0,1,1)
A_4A_5	(1,1,1,0)(1,2,1,0)(1,1,2,0)(1,1,1,1)	$A_{13}A_{12}$	(0,1,1,1)(1,1,1,1)
A_4A_6	(1,1,1,0))(0,1,1,0)	$A_{12}A_{15}$	(1,0,1,1)(1,1,1,1)(1,0,1,2)(1,0,2,1)
A_4A_{13}	(0,1,1,1)(1,1,1,1)	$A_{13}A_{14}$	(1,1,1,1)(1,2,1,1)
A_7A_5	(1,1,1,0)(2,1,1,0)	$A_{14}A_{15}$	(1,1,1,1)(2,1,1,1)

From Table 2 it follows that the really neighboring 4 - quadrangular prisms, when translating them to the length of the edge along one of the coordinates, share quadrangular and triangular flat faces and edges with common elements. When translating over the length of an edge along several coordinates, as follows from analytical Expressions (4), the dimension of common elements is even smaller. And when broadcasting on two or more edge lengths, common elements of 4 - quadrangular prisms are absent.

Since in the vicinity of any 4 - quadrangular prisms at a distance of the length of the edge along each coordinate there is another 4 - quadrangular prisms having a common faces with a central 4 - quadrangular prisms, and on the diagonal of two coordinates there is a 4 - quadrangular prisms having a common face with the central 4 - quadrangular prisms, then the 4 - quadrangular prisms during translation fills the entire four - dimensional space without gaps. This means that the 4 - quadrangular prisms is a four - dimensional parallelohedron.

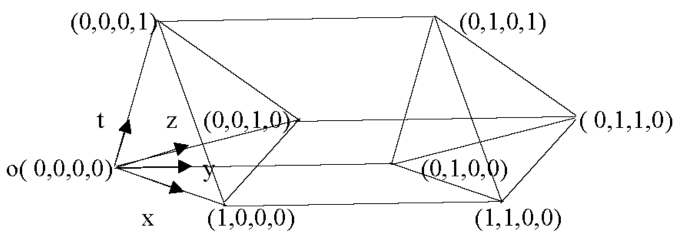
Partition of Space by the Polytopic Prism

One of the simplest types of the polytopic prismahedrons is the tetrahedral prism, i.e. a product tetrahedron by the segment (Zhizhin, 2019 a). Its structural formula has form:

$$P_4^3(4F_3^2)\times P_2^1 = P_8^4[4P_6^3(3F_4^2,2F_3^2),2F_4^3(4F_3^2)] \quad (5)$$

Here, the subscript in the polytope P and his face F indicates the number of vertices, and the superscript indicates the dimension of the corresponding polytope or faces. The right side of (4) describing the structural formula of the product, the facet indicated by the symbol of the polytope to specify which polytopes of dimension n - 1 is composed work polytopes. Thus, P_4^3 - the tetrahedron, P_2^1 - the segment, P_6^3 - the triangular prism, P_3^2 - the triangle, P_4^2 - the quadrilateral, P_8^4 - the tetrahedral prism. The dimension of the tetrahedral prism is equal to 4, it has 8 vertices, 16 edges, 14 faces two - dimensional, 6 three - dimensional faces (2 tetrahedrons, 4 triangular prisms). Image tetrahedral prism is shown in Figure 10.

Figure 10. The tetrahedral prism



Can introduce one of the vertices of the tetrahedral prism origin of the four - dimensional space (x, y, z, t). Orient the coordinates, such as indicated in Figure 10.

Assume that the length of each edge is equal to 1. Then, each node tetrahedral prism can be associated with a set of integers (Figure 10). Translating tetrahedral prism along the coordinates x, y, z, t, we obtain the lattice vertices. Let A_0 tetrahedral prism with the values of vertex coordinates in Figure 10. Then:

$$A_0 = (0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1) \qquad (6)$$

$A_1 = A_0(x+1) = (1, 0, 0, 0), (2, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (2, 1, 0, 0),$
$(1, 1, 1, 0), (1, 1, 0, 1),$

$A_2 = A_0(z+1) = (0, 0, 1, 0), (1, 0, 1, 0), (0, 1, 1, 0), (0, 0, 2, 0), (0, 0, 1, 1), (1, 1, 1, 0),$
$(0, 1, 2, 0), (0, 1, 1, 1),$

$A_3 = A_0(x+1, z+1) = A_1(z+1) = (1, 0, 1, 0), (2, 0, 1, 0), (1, 1, 1, 0), (1, 0, 2, 0), (1, 0, 1, 1),$
$(2, 1, 1, 0), (1, 1, 2, 0), (1, 1, 1, 1),$

$A_4 = A_0(y+1, z+1) = A_2(y+1) = (0, 1, 1, 0), (1, 1, 1, 0), (0, 2, 1, 0), (0, 1, 2, 0), (0, 1, 1, 1),$
$(1, 2, 1, 0), (0, 2, 2, 0), (0, 2, 1, 1),$

$A_5 = A_0(x+1, y+1, z+1) = A_3(y+1) = (1, 1, 1, 0), (2, 1, 1, 0), (1, 2, 1, 0), (1, 1, 2, 0), (1, 1, 1, 1),$
$(2, 2, 1, 0), (1, 2, 2, 0), (1, 2, 1, 1),$

$A_6 = A_0(y+1) =$ (0, 1, 0, 0), (1, 1, 0, 0), (0, 2, 0, 0), (0, 1, 1, 0), (0, 1, 0, 1), (1, 2, 0, 0), (0, 2, 1, 0), (0, 2, 0, 1),

$A_7 = A_0(y+1, x+1) = A_1(y+1) =$ (1, 1, 0, 0), (2, 1, 0, 0), (1, 2, 0, 0), (1, 1, 1, 0), (1, 1, 0, 1), (2, 2, 0, 0), (1, 2, 1, 0), (1, 2, 0, 1),

$A_8 = A_0(t+1) =$ (0, 0, 0, 1), (1, 0, 0, 1), (0, 1, 0, 1), (0, 0, 1, 1), (0, 0, 0, 2), (1, 1, 0, 1), (0, 1, 1, 1), (0, 1, 0, 2),

$A_9 = A_0(y+1, t+1) = A_6(t+1) =$ (0, 1, 0, 1), (1, 1, 0, 1), (0, 2, 0, 1), (0, 1, 1, 1), (0, 1, 0, 2), (1, 2, 0, 1), (0, 2, 1, 1), (0, 2, 0, 2),

$A_{10} = A_0(x+1, y+1, t+1) = A_7(t+1) =$ (1, 1, 0, 1), (2, 1, 0, 1), (1, 2, 0, 1), (1, 1, 1, 1), (1, 1, 0, 2), (2, 2, 0, 1), (1, 2, 1, 1), (1, 2, 0, 2),

$A_{11} = A_0(x+1, t+1) = A_1(t+1) =$ (1, 0, 0, 1), (2, 0, 0, 1), (1, 1, 0, 1), (1, 0, 1, 1), (1, 0, 0, 2), (2, 1, 0, 1), (1, 1, 1, 1), (1, 1, 0, 2),

$A_{12} = A_0(y+1, z+1) = A_2(t+1) =$ (0, 0, 1, 1), (1, 0, 1, 1), (0, 1, 1, 1), (0, 0, 2, 1), (0, 0, 1, 2), (1, 1, 1, 1), (0, 1, 2, 1), (0, 1, 1, 2),

$A_{13} = A_0(z+1, y+1, t+1) = A_4(t+1) =$ (0, 1, 1, 1), (1, 1, 1, 1), (0, 2, 1, 1), (0, 1, 2, 1), (0, 2, 2, 1), (0, 1, 1, 2), (1, 2, 1, 1), (0, 2, 1, 2),

$A_{14} = A_0(x+1, y+1, z+1, t+1) = A_5(t+1) =$ (1, 1, 1, 1), (2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2), (2, 2, 1, 1), (1, 2, 2, 1), (1, 2, 1, 2),

$A_{15} = A_0(x+1, t+1, z+1) = A_3(t+1) =$ (1, 0, 1, 1), (2, 0, 1, 1), (1, 1, 1, 1), (1, 0, 2, 1), (1, 0, 1, 2), (2, 1, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2)

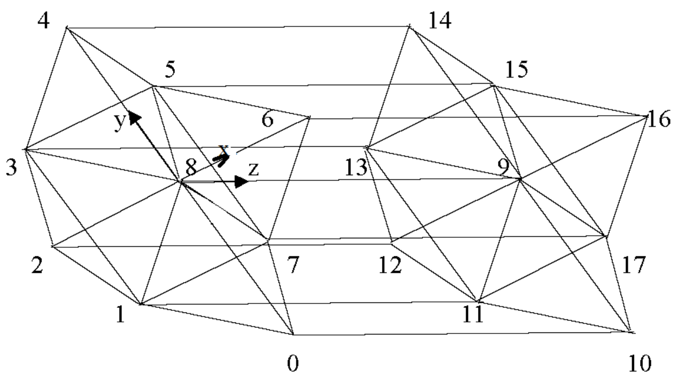
Representing the tetrahedral prisms $A_0 \div A_{15}$ dots in four - dimensional space, to get the hypercube. Moreover, the edges of the hypercube correspond to possible changes in the values of one of the coordinates of the vertices of the tetrahedral prism unit ((4 - cube A2, Figure 8, in which the polytopes at vertices correspondent the equality (6)).

In addition, each edge of the hypercube in Figure 8 in this case can be considered as an element of the overall two tetrahedral prisms, connected by an edge. Using the coordinate expression tetrahedral prisms (6) can be analytically determined. In Table 3 geometry elements are common to each pair of the tetrahedral prisms connected by an edge in Figure 8 are listed.

Diagonals flat faces in the hypercube correspond to a simultaneous change in the unit values of the two coordinates of the vertices of the tetrahedral prisms (Table 3). Diagonals 8 cubes in the hypercube in Figure 8 correspond to a simultaneous change in the unit of some three coordinates. Common elements of the tetrahedral prisms with such change vertex coordinates is either a vertex or the empty set. When you change the same unit coordinates all four common elements in the tetrahedral prisms not.

Between A_i type tetrahedral prisms arranged tetrahedral prism type B_i (see Figure 11) having a closest to them tetrahedral prisms A_i general quadrangular two - dimensional face.

Figure 11. The complex of the tetrahedral prisms



The tetrahedral prisms A_i and B_i have common edges, which are parties to the general quadrilateral faces. The tetrahedral prisms A_i and B_i are connected symmetry transformation - turning the 180^0 around a common edge of the tetrahedron. Forms A_i and B_i are two enantiomorphic forms of a tetrahedral prism. Let us prove that the complex of two pairs of enantiomorphic forms of tetrahedral prisms (Figure 11) has dimension 5. As follows from Figure 11, this complex is not a convex polytope. However, for its analysis, the Euler – Poincare equation can be used, since Poincare showed that this equation is applicable to all connected forms (Poincare, 1895).

The polytope in Figure 11 has 18 vertices, $f_0 = 18$. The polytope in Figure 11 has 47 edges:

0 – 1, 0 – 8, 0 – 7, 0 – 10, 1 – 7, 1 – 8, 1 – 3, 1 – 2, 1 – 11, 2 – 3, 2 – 8, 2 – 12, 3 – 8, 3 – 5, 3 – 4, 3 – 13, 4 – 8, 4 – 5, 4 – 14, 5 – 8, 5 – 6, 5 – 15, 6 – 8, 6 – 7, 6 – 16, 7 – 8, 7 – 17, 8 – 9, 9 – 10, 9 – 11, 9 – 12, 9 – 13, 9 – 14, 9 – 15, 9 – 16, 9 – 17, 10 – 11, 10 – 17, 11 – 12, 11 – 13, 12 – 13, 13 – 14, 13 – 15, 14 – 15, 15 – 16, 15 – 17, 16 – 17, i.e. $f_1 = 47$

The polytope in Figure 11 has 50 two – dimensional elements:

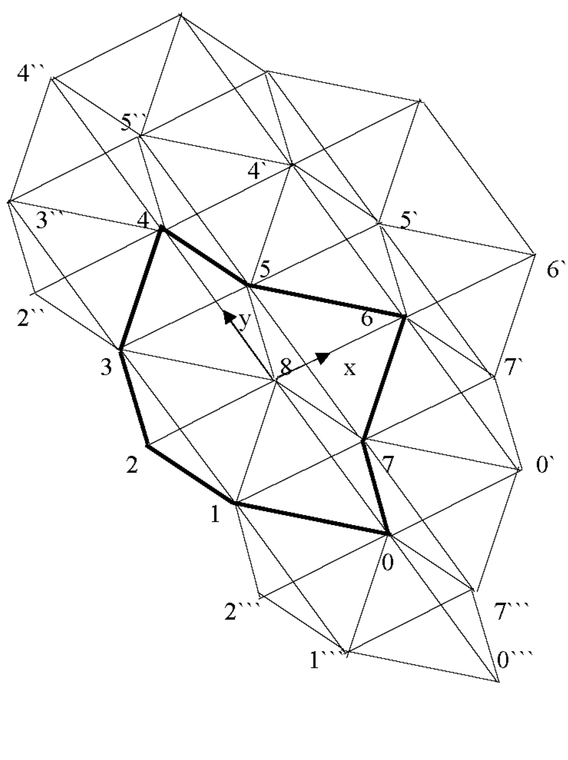
0 – 8 – 9 – 10, 0 – 7 – 17 – 10, 0 – 1 – 11 – 10, 0 – 8 – 7, 0 – 1 – 8, 0 – 1 – 7, 1 – 8 – 7, 1 – 2 – 8, 1 – 2 – 3, 1 – 3 – 8, 1 – 2 – 12 – 11, 1 – 8 – 9 – 11, 1 – 17 – 7 – 11, 1 – 3 – 11 – 13, 2 – 3 – 13 – 12, 2 – 3 – 9 – 12, 2 – 3 – 8, 3 – 5 – 15 – 13, 4 – 3 – 14 – 13, 3 – 4 – 5, 3 – 5 – 8, 3 – 4 – 8, 4 – 5 – 8, 4 – 14 – 8 – 9, 4 – 5 – 15 – 14, 5 – 6 – 7, 5 – 8 – 7, 5 – 6 – 8, 5 – 15 – 8 – 9, 5 – 17 – 7 – 15, 5 – 6 – 15 – 16, 6 – 7 – 8, 6 – 16 – 7 – 17, 6 – 16 – 8 – 9, 7 – 8 – 9 – 17, 9 – 13 – 14, 9 – 13 – 15, 9 – 14 – 15, 9 – 15 – 16, 9 – 15 – 17, 9 – 16 – 17, 9 – 17 – 10, 9 – 10 – 11, 9 – 11- 17, 10 – 17 – 11, 11 – 12 – 13, 13 – 14 – 15, 15 – 10 – 17, 3 – 5 – 7 – 1,11 – 13 – 15 – 17, i.e. $f_2 = 50$

The polytope in Figure 11 has 25 three – dimensional elements:

3 – 4 – 5 – 8, 14 – 15 – 13 – 9, 5 – 6 – 7 – 8,15 – 16 – 17 – 9, 1 – 2 – 3 – 8, 11 – 12 – 13 – 9, 8 – 7 – 0 – 1, 10 – 11 – 17 – 9, 0 – 8 – 7 – 10 – 17 – 9, 0 – 1 – 7 – 10 – 11 – 17, 0 – 1 – 8 – 10 – 11 – 9, 1 – 8 – 7 – 11 – 9 – 17, 1 – 2 8 – 11 – 12 – 9, 1 – 2 – 3 – 13 – 12 – 11, 2 – 3 – 8 – 12 – 13 – 9, 3 – 5 – 8 – 13 – 15 – 9, 3 – 4 – 5 – 13 – 14 – 15, 4 – 5 – 8 – 14 – 15 – 9, 5 – 8 – 6 – 15 – 16 – 9, 5 – 6 – 7 – 15 – 16 – 17, 5 – 8 – 7 – 15 – 9 – 17, 3 – 5 – 1 – 7 – 8, 13 – 15 – 17 – 11 – 9, 3 – 5 – 7 – 1 – 13 – 15 – 17 – 11, 1 – 2 – 3 – 4 – 5 – 6 – 7 – 8 – 9 – 10 – 11 – 12 – 13 – 14 – 15 – 16 – 17 – 18 (three - dimensional body bounded by the outer surface of the complex), i.e. $f_3 = 25$

The polytope in Figure 11 has 6 four – dimensional elements: four of the tetrahedral prisms and two bases of the complex, each of which consists of four tetrahedrons. Consider one of these bases (Figure 12).

Figure 12. The base of the tetrahedral prism complex



It has 9 vertices, i.e. $f_0 = 9$, 20 edges:

0 – 1, 1 – 2, 2 – 3, 3 – 4, 4 – 5, 5 – 6, 6 – 7, 0 – 7, 8 – 3, 8 – 4, 8 – 5, 8 – 6, 8 – 7, 8 – 0, 8 – 2, 8 – 1, 3 – 5, 1 – 7, 3 – 1, 5 – 7, $f_1 = 20$

The complex from four tetrahedrons has 17 two – dimensional faces:

3 – 4 – 5, 3 – 5 – 8, 3 – 4 – 8, 3 – 8 – 2, 3 – 8 – 1, 3 – 1 – 2, 4 – 5 – 8, 5 – 6 – 8, 5 – 6 – 7, 5 – 8 – 7, 6 – 7 – 8, 1 – 7 – 8, 1 – 7 – 0, 1 – 8 – 0, 1 – 8 – 2, 1 – 3 – 5 – 7, i.e. $f_2 = 17$

The complex from four tetrahedrons has 6 three – dimensional faces.

They are four tetrahedrons, one pyramid 3 – 5 – 7 – 1 – 8 and three - dimensional body bounded by the outer surface of the complex, i.e. $f_3 = 6$. Substituting the obtained values into the Euler - Poincare Equation (2), we obtain 9 – 20 + 17 – 6 = 0. The Euler - Poincaré equation for the base of the tetrahedral prism complex is satisfied for a dimension equal to 4, therefore this base has dimension 4. Then for the entire complex of tetrahedral prisms in Figure 11, the number of elements with dimension 4 is 6, i.e. $f_4 = 6$. Substituting the obtained values for the complex of tetrahedral prisms in the Euler-Poincare Equation (2), we obtain:

$$18 - 47 - 50 - 25 + 6 = 2$$

This means that in this case the Euler - Poincaré equation is satisfied for a dimension equal to 5, i.e. the tetrahedral prism complex in Figure 11 has dimension 5. It should be noted that not all edges in the complex of tetrahedral prisms form continuous lines, taking into account their possible continuation in neighboring tetrahedral prisms. Therefore, in this case, when choosing a coordinate system, it is necessary to select as the directions of the coordinate variables those edges that form continuous straight lines. There are three such directions, as can be seen from Figure 11. This is the direction along the generators of the complex of tetrahedral prisms (z) and two directions perpendicular to these generators (x, y) (Figure 11). Assuming that the length of the edges of the complex of tetrahedral prisms is equal to unity, we assign coordinates (x, y, z) to its vertices:

0(0, -1, 0), 1(-1/2, -1/2, 0), 2(-1, 0, 0), 3(-1/2, 1/2, 0), 4(0, 1, 0),
5(1/2, 1/2, 0), 6(1, 0, 0), 7(1/2, -1/2, 0), 8(0, 0, 0), 9(0, 0, 1) (7)
10(0, -1, 1), 11(-1/2, -1/2, 1),12(-1, 0, 1), 13(-1/2, 1/2, 1),
14(0, 1, 1), 15(1/2, 1/2, 1), 16(1, 0, 1), 17(1/2, -1/2, 1)

The distribution of tetrahedral prisms B_i can be determined in the same way as the distribution of tetrahedral prisms A_i (see Zhizhin, 2019 a). However, in this case, of great interest is the question of the spatial motion of the entire complex of tetrahedral prisms, consisting of two pairs of enantiomorphic tetrahedral prisms.

Theorem 2: The complex, consisting of two pairs of enantiomorphic tetrahedral prisms adjacent to each other over whole quadrangular plane sections, is a parallelohedron of dimension 5.

Proof: The fact that the complex consisting of two pairs of enantiomorphic tetrahedral prisms has dimension 5 has already been proved. It remains to prove that it is a parallelohedron. For this, we assume that the initial state of the complex is that which is determined by the values of its vertices (9). Let us depict one of the bases (for example, the left) of the tetrahedral prisms complex in a separate figure in bold line segments.

We will not depict the generators of the tetrahedral prism complex so as not to clutter up the pattern. We remember that the generators come from each vertex of the base and remain parallel to themselves and the axis z. We will not depict the right base either, since it is parallel to the left base. We increase the values of the coordinate x by one step (one rib length) at all the vertices of the left base of the tetrahedral prism complex. This will correspond to the translation of the complex of tetrahedral prisms by one step along the *x* axis. Then the following mapping of the vertices of the left base of the complex:

0(0, -1, 0) → 0`(1, -1, 0), 1(-1/2, -1/2, 0) → 7(1/2, -1/2, 0), 2(-1, 0, 0) → 8(0, 0, 0),
3(-1/2, 1/2, 0) → 5(1/2, 1/2, 0), 4(0, 1, 0) → 4`(1, 1, 0), 5(1/2, 1/2, 0) → 5`(3/2, 1/2, 0),
6(1, 0, 0) → 6`(2, 0, 0), 7(1/2, -1/2, 0) → 7`(3/2,-1/2, 0), 8(0, 0, 0) → 6(1, 0, 0)

As a result of the mapping, a new base of the complex of tetrahedral prisms was formed 0`- 7 – 8 – 5 – 4` - 5` - 6` - 7` - 6. This base has dimension 4 and common tetrahedron 7 – 8 – 5 – 6 with initial left base.

Now we increase the values of the coordinate *y* by one step (one rib length) at all the vertices of the left base of the tetrahedral prism complex. This will correspond to the translation of the complex of tetrahedral prisms by one step along the *y* axis. Then the following mapping of the vertices of the left base of the complex:

0(0, -1, 0) → 8(0, 0, 0), 1(-1/2, -1/2, 0) → 3(-1/2, 1/2, 0), 2(-1, 0, 0) → 2``(-1, 1, 0),
3(-1/2, 1/2, 0) → 3``(-1/2, 3/2, 0), 4(0, 1, 0) → 4``(0, 2, 0), 5(1/2, 1/2, 0) → 5``(1/2, 3/2, 0),
6(1, 0, 0) → 4`(1, 1, 0), 7(1/2, -1/2, 0) → 5(1/2, 1/2, 0), 8(0, 0, 0) → 4(0, 1, 0)

As a result of the mapping, a new base of the complex of tetrahedral prisms was formed 8 – 3 – 2`` – 3`` – 4`` - 5`` - 4` - 5 - 4. This base has dimension 4 and common tetrahedron 3 – 8 – 5 – 4 with initial left base.

Now we decrease the values of the coordinate *y* by one step (one rib length) at all the vertices of the initial left base of the tetrahedral prism complex. This will correspond to the translation of the complex of tetrahedral prisms by one step opposite the *y* axis. Then the following mapping of the vertices of the left base of the complex:

0(0, -1, 0) → 0```(0, -2, 0), 1(-1/2, -1/2, 0) → 1```(-1/2,-3/2, 0), 2(-1, 0, 0) → 2```(-1, -1, 0),
3(-1/2, 1/2, 0) → 1(-1/2, -1/2, 0), 4(0, 1, 0) → 8(0, 0, 0), 5(1/2, 1/2, 0) → 7(1/2, -1/2, 0),
6(1, 0, 0) → 0`(1, -1, 0), 7(1/2, -1/2, 0) → 7```(1/2,-3/2, 0), 8(0, 0, 0) → 0(0, -1, 0).

As a result of the mapping, a new base of the complex of tetrahedral prisms was formed 0```– 1``` – 2``` – 1 – 8 - 7 - 0` - 7``` - 0. This base has dimension 4 and common tetrahedron 1 – 8 – 7 – 0 with initial left base.

Similar broadcasts in the coordinate *x* and coordinate *y* can be continued. In all cases, we will receive new instances of the left bases of the tetrahedral prism complex. Moreover, each of these bases will have a common tetrahedron with any of the neighboring bases. Now recall that from each vertex of the base of the tetrahedral prism complex, the generators of the tetrahedral prism complex emanate. Therefore, the translation of the left base of the complex along the coordinates *x* and *y* corresponds to the translation of the entire complex of tetrahedral prisms along these coordinates. In these translations of the tetrahedral prism complex, the common between neighboring complexes will no longer be tetrahedrons, but tetrahedral prisms of dimension 4. In addition, there is still the possibility of translating the complex of tetrahedral prisms along the coordinate *z*. With this translation (one step), the left base of the complex takes the place of the right base of the initial position of the complex. Each of them has dimension 4. From the analysis it follows that the translation of the five - dimensional complex of tetrahedral prisms leads to a complete filling of the space without gaps with these complexes, which are in a parallel position relative to each other. This proves that the five - dimensional complex of tetrahedral prisms is a parallelohedron. Q.E.D.

CONCLUSION

Due to the fact that in the author's works (Zhizhin, 2013, 2014, 2018, 2019 a) the existence of a higher dimension in the nanoworld was established, the task of constructing n - dimensional space from n - dimensional figures has become urgent. The solution to this problem is necessary in the technology of obtaining new nanomaterials. For almost 100 years, many scientists have tried to solve the problem of constructing n - dimensional spaces by mechanically transferring the properties of three - dimensional space to multidimensional spaces. This was the basis of the Delaunay theory, which is still actively promoted as a means of solving this problem. Moreover, real results in the field of higher dimension in this theory have not yet been obtained. In this paper, it is proved that the real parallelohedrons of the highest dimension, from which the space of the highest dimension can be built, are polytopic prismahedrons or figures composed of them. This is demonstrated by a number of relatively simple examples. Using the classification of polytopic prismahedrons (Zhizhin, 2018, 2019a), various high - dimensional parallelohedrons can be obtained for the construction of new nanomaterials.

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