Appendix:
Fourier Analysis and Total Harmonic Distortion (THD) of Waveforms

For the purposes of the harmonic analysis of some functions, characterizing the basic variables of the power electronic converters, at first an auxiliary function shown in Fig.A.1 is examined. This allows using a unified approach appropriate for the basic waveforms of some variables at these converters.

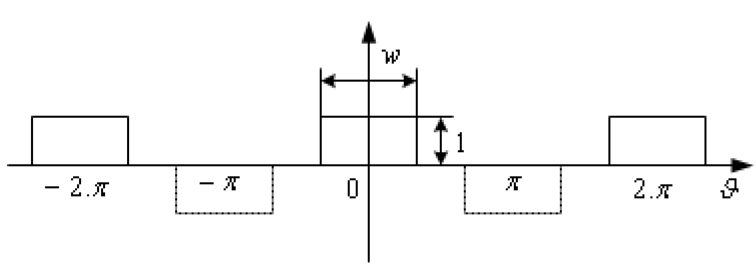
The auxiliary function consists of positive and negative pulses with amplitude 1 and duration *w*. If the beginning of the coordinate system is the middle of a positive pulses, then the coefficients of Fourier series for the positive pulses are:

$$A_0 = \frac{1}{2\pi}\int_{-\frac{w}{2}}^{\frac{w}{2}} 1.d\vartheta = \frac{w}{2.\pi} \qquad \text{(A.1)}$$

$$A_k = \frac{1}{\pi}\int_{-\frac{w}{2}}^{\frac{w}{2}} 1.cos\left(k.\vartheta\right).d\,\vartheta = \frac{2}{k.\pi} sin\frac{k.w}{2} \qquad \text{(A.2)}$$

Also, the Fourier series for the positive pulses is:

Figure A.1. Auxiliary function waveform



$$F_P = \frac{2}{\pi}\left(\frac{w}{4} + sin\frac{w}{2}cos\,\vartheta + \frac{1}{2}sin\frac{2.w}{2}cos\,2.\,\vartheta + \frac{1}{3}sin\frac{3.w}{2}cos\,3.\,\vartheta + \frac{1}{4}sin\frac{4.w}{2}cos\,4.\,\vartheta + \frac{1}{5}sin\frac{5.w}{2}cos\,5.\,\vartheta ...\right) \quad (A.3)$$

Analogically, the Fourier series for the negative pulses is found, and, it is:

$$F_P = \frac{2}{\pi}\left(-\frac{w}{4} + sin\frac{w}{2}cos\,\vartheta - \frac{1}{2}sin\frac{2.w}{2}cos\,2.\,\vartheta + \frac{1}{3}sin\frac{3.w}{2}cos\,3.\,\vartheta - \frac{1}{4}sin\frac{4.w}{2}cos\,4.\,\vartheta + \frac{1}{5}sin\frac{5.w}{2}cos\,5.\,\vartheta ...\right) \quad (A.4)$$

Summing (A.3) and (A.4), the Fourier series for the auxiliary function is found as:

$$F = \frac{4}{\pi}\left(sin\frac{w}{2}cos\,\vartheta + \frac{1}{3}sin\frac{3.w}{2}cos\,3.\,\vartheta + \frac{1}{5}sin\frac{5.w}{2}cos\,5.\,\vartheta + ...\right) \quad (A.5)$$

Fig.A.2 depicts three waveforms typical for the power electronic converters.

Using equation (A.5), the harmonic content of the three functions may be determined consistently.

For Fig.A.2.a – taking in consideration that the pulse amplitude is F, and their duration is $w = \pi$

$$f = \frac{4.F}{\pi}.\left(cos\,\vartheta - \frac{1}{3}cos\,3.\,\vartheta + \frac{1}{5}cos\,5.\,\vartheta - ...\right) \quad (A.6)$$

Figure A.2. Waveforms typical for the power electronic converters, a) source current in the rectifiers shown in Chapter 4, Figure 15 and 17; output voltage in single-phase uncontrolled inverter – Chapter 7, Figure 6. b) source current in the rectifiers shown in Chapter 4, Figure 16 and Figure 18; output voltage in single-phase controlled inverter – Chapter 7, Figure 6. c) source current for a phase in the rectifiers shown Chapter 4, Figure 21; phase-to-phase voltage in three-phase inverter – Chapter 7, Figure 18.

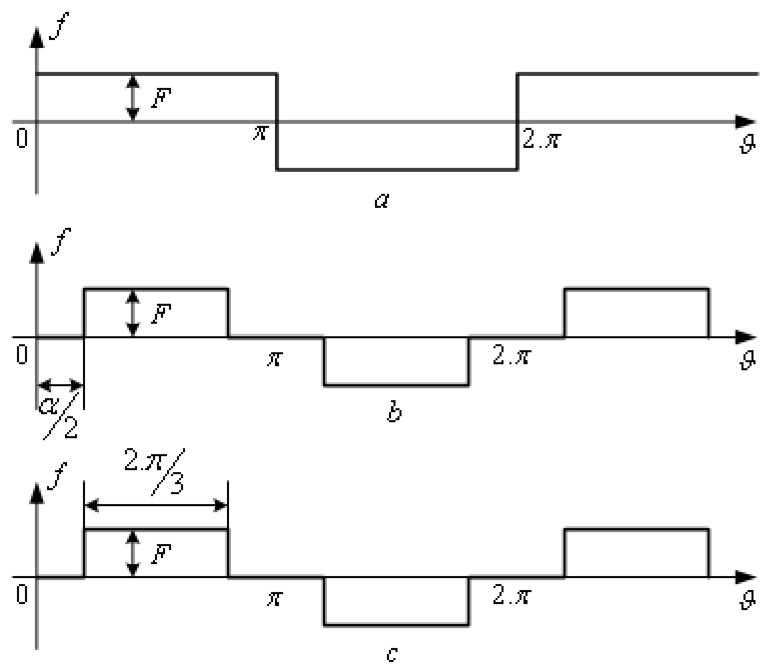


Fig.A.3 shows the distribution of the most important harmonics.

Equation (A.6) shows the relationship between the values of the higher harmonics and this of the first harmonic as:

$$\frac{F_k}{F_1} = \frac{1}{k} \tag{A.7}$$

For Fig.A.2.b - taking in consideration that the pulse amplitude is *F*, and their duration is $w = \pi - \alpha$

$$f = \frac{4.F}{\pi}\left(sin\frac{\pi - \alpha}{2} cos\vartheta + \frac{1}{3} sin\frac{3.(\pi - \alpha)}{2} cos 3.\vartheta + \frac{1}{5} sin\frac{5.(\pi - \alpha)}{2} cos 5.\vartheta + ...\right) \tag{A.8}$$

The numbers of the harmonics are the same as in the previous case but their percentage content is dependent on the firing angle α . The relationship between the value of the kth higher harmonic and this of the first harmonics is:

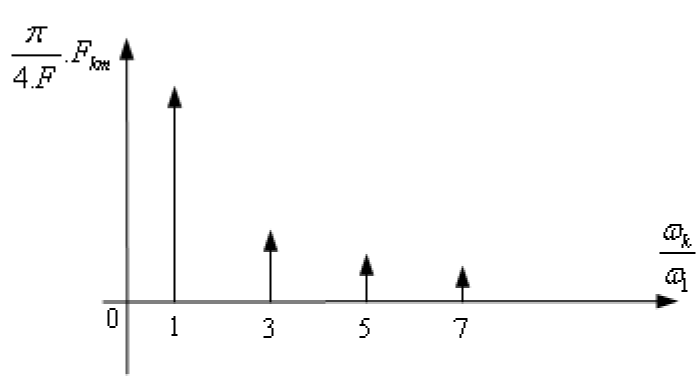
$$\frac{F_k}{F_1} = \frac{1}{k}.\frac{\sin\frac{k.(\pi - \alpha)}{2}}{\sin\frac{\pi - \alpha}{2}} \tag{A.9}$$

For Fig.A.2.c - taking in consideration that the pulse amplitude is *F*, and their duration is $w = \frac{2.\pi}{3}$

$$f = \frac{2.\sqrt{3}.F}{\pi}.\left(cos\vartheta - \frac{1}{5} cos 5.\vartheta + \frac{1}{7} cos 7.\vartheta - \frac{1}{11} cos 11.\vartheta + \frac{1}{13} cos 13.\vartheta ...\right) \tag{A.10}$$

For this case, Fig.A.4 depicts the distribution of the most important harmonics.

Figure A.3. Distribution of harmonics of the function shown in Figure A.2.a



Equation (A.10) shows that here again the relationship between the values of the higher harmonics and this of the first harmonic is valid found as (A.7).

The equations found may be used to find the harmonic coefficient K_H (see (4.7), which is also called "total harmonic distortion (THD)", for the examined typical functions.

At first, this is made for the function shown in Fig.A.2.b. For this purpose, the effective value of the function is found as:

$$F_{RMS} = \sqrt{\frac{1}{\pi}\int_0^{\pi-\alpha} F^2.d\vartheta} = F\sqrt{1-\frac{\alpha}{\pi}} \qquad \text{(A.11)}$$

The effective value f the first harmonic using (A.8) is:

$$F_1 = \frac{2.\sqrt{2}}{\pi}.F.\sin\frac{\pi-\alpha}{2} \qquad \text{(A.12)}$$

The distortion coefficient for the current waveform is found form (A.11) and (A.12):

$$\nu = \frac{F_1}{F_{RMS}} = \frac{2.\sqrt{2}}{\sqrt{\pi.(\pi-\alpha)}}.\sin\frac{\pi-\alpha}{2} \qquad \text{(A.13)}$$

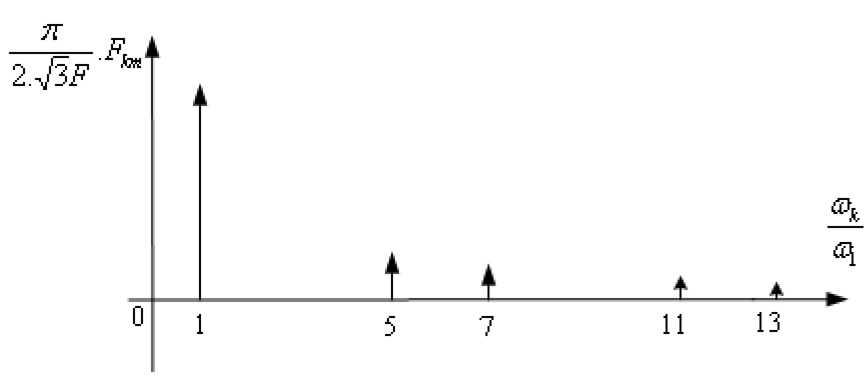
Using the equation (4.8), K_H is derived as:

$$K_H \equiv THD = \frac{1}{\nu}.\sqrt{1-\nu^2} = 100.\sqrt{\frac{\pi.(\pi-\alpha)}{8.\sin^2\left(\frac{\pi-\alpha}{2}\right)}-1}, \quad \% \qquad \text{(A.14)}$$

An analysis of function (A.14) for an extremium may be performed and the minimum value of K_H is found to be 29% at $\alpha = 46.44^{\circ}$.

Using (A.14), the value of K_H for the function shown in Fig.A.2.a at $\alpha = 0$ is found to be:

Figure A.4. Distribution of harmonics of the function shown in Figure A.2.c



$$K_H = 100.\sqrt{\frac{\pi^2}{8} - 1} = 48.34\% \quad \text{(A.15)}$$

For the function shown in Fig.A.2.c $\alpha = \frac{\pi}{3}$, and again using (A.14), it is found:

$$K_H = 100.\sqrt{\frac{\pi^2}{9} - 1} = 31.08\% \tag{A.16}$$

Thus, equations for Fourier series and for harmonic coefficient have been found for the most typical waveforms of the power electronic converters.

In the cases of some more complicated waveforms of certain variables (for example, sinusoidal pulse-width modulation, selective elimination of harmonics see Chapter 7, Figures 9, 11, 13, 22), as well as in the case of several leveled modulation (see Chapter 7, Figure 16), a particular approach is applied in solving the above-stated topics.

Only an example will be examined here – sinusoidal pulse-width modulation shown in Fig.A.5. The waveform shown is typical for a single-phase voltage inverter (see Chapter 5, Figure 10) and also for the source current of the single-phase rectifier(see Chapter 4, Figure 41.b).

The duration of the i-th conductivity interval is defined from the equation:

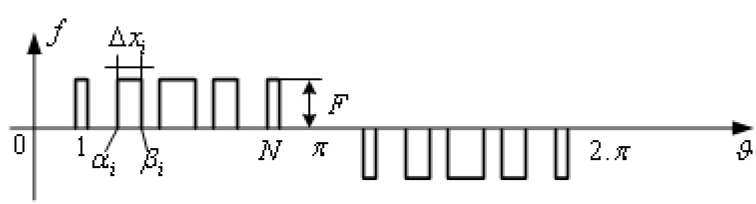
$$\Delta x_i = \frac{\pi}{N+1} \cdot M \cdot \sin\left(i \cdot \frac{\pi}{N+1}\right); \quad i = 1, 2, \ldots N, \tag{A.17}$$

where in $0 \le M \le 1$ is a modulation coefficient.

The turn on and off angles of the switches are:

$$\begin{aligned} \alpha_i &= i \cdot \frac{\pi}{N+1} - \frac{1}{2} \cdot \frac{\pi}{N+1} \cdot M \cdot \sin\left(i \cdot \frac{\pi}{N+1}\right); \quad i = 1, 2, \ldots N \\ \beta_i &= i \cdot \frac{\pi}{N+1} + \frac{1}{2} \cdot \frac{\pi}{N+1} \cdot M \cdot \sin\left(i \cdot \frac{\pi}{N+1}\right); \quad i = 1, 2, \ldots N \end{aligned} \tag{A.18}$$

Figure A.5. Sinusoidal pulse-width modulation



Using Figure A.5, the maximum value of the k-th harmonic may be determined:

$$F_{km} = \frac{2}{\pi} = \sum_{i=1}^{N} \int_{\alpha_i}^{\beta_i} F.sin(n\vartheta).d\,\vartheta \tag{A.19}$$

After solving (A.19), the following equation is found:

$$\frac{F_{km}}{F} = \frac{2}{n.\pi} \sum_{i=1}^{N} \left(\cos n.\alpha_i - \cos n.\beta_i \right) \tag{A.20}$$

After substitution using (A.18), an equation for the relative maximum value of the k-th harmonic is found as:

$$\frac{\pi.F_{km}}{4.F} = \frac{1}{n} \sum_{i=1}^{N} \sin\left(i\frac{n.\pi}{N+1} \right).\sin\left[\frac{n.\pi}{2.(N+1)}.M.\sin\left(i\frac{\pi}{N+1} \right) \right] \tag{A.21}$$

At linear pulse-width modulation (see Chapter 4, Figure 41.a), the relationships already found may be uses, having in mind that:

$$\Delta x_i = \Delta x = \frac{\pi}{N+1} \cdot M; \quad i = 1, 2, \ldots N \tag{A.22}$$

Therefore:

$$\alpha_i = \frac{\pi}{N+1} \cdot \left(i - \frac{M}{2} \right); \quad i = 1, 2, \ldots N$$
$$\beta_i = \frac{\pi}{N+1} \cdot \left(i + \frac{M}{2} \right); \quad i = 1, 2, \ldots N \tag{A.23}$$

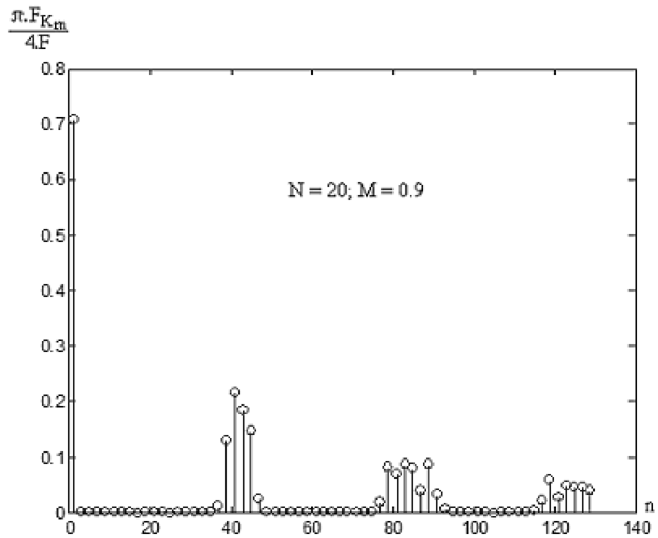
After substituting in (A.20), it is found:

$$\frac{\pi.F_{km}}{4.F} = \frac{1}{n} \sum_{i=1}^{N} \sin\left(i\frac{n.\pi}{N+1} \right).\sin\left(\frac{n.\pi}{2.(N+1)}.M \right) \tag{A.24}$$

Figure A.6 depicts the distribution of the harmonics at sinusoidal pulse-width modulation corresponding to the equation (A.21), while Fig.A.7 – at linear pulse-width modulation corresponding to the equation (A.24). The graphics are obtained using *MATLAB*[1] (see Chapter 3). The relationships shown are made for $N = 20$, $M = 0.9$.

The following example illustrates the possibility to made harmonic analysis using *ORCAD*[2] simulation software. The operation of a single-phase voltage inverter controlled according to the method shown in Chapter 7, Figure 11 is simulated. The schematic for the computer simulation is shown in Fig.A.8. The effective value gained of the first harmonic of the output voltage is 230V at active-inductive load of 10Ω, 10mH. The frequency of the saw voltage is 1 kHz (see Chapter 7, Figures 11 and 12). The ratio of the amplitude of the reference sinusoidal waveforms to the amplitude of the saw voltage is 0.9. These

Figure A.6. Distribution of the harmonics at sinusoidal pulse-width modulation



data correspond to the output data used to gain the graphical relationship shown in Figure A.6 using *MATLAB*[1].

The waveforms of the inverter output voltage, as well as the sinusoidal voltage across the load, are shown in Figure A.9.

Figure A.10 depicts the harmonic spectrum of the inverter output voltage.

Figure A.7. Distribution of the harmonics at linear pulse-width modulation

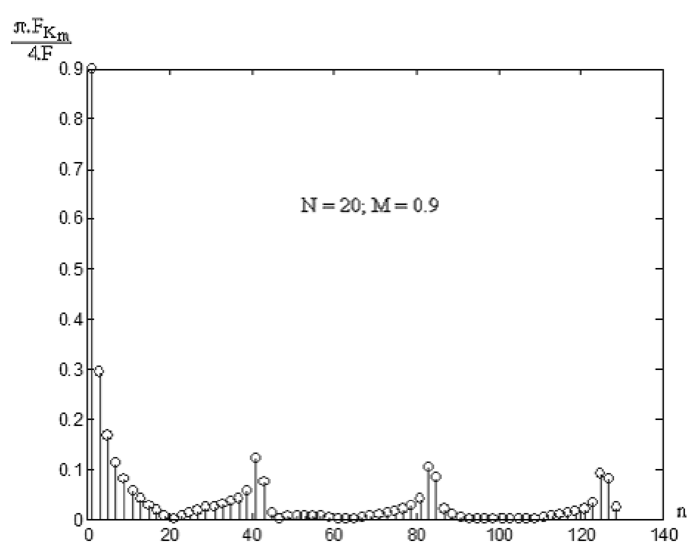
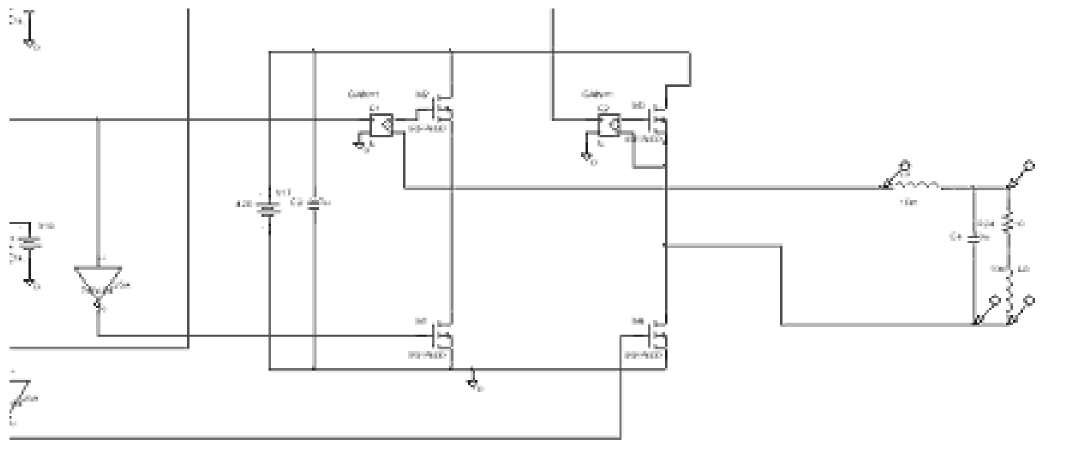


Figure A.8. Computer simulation schema



ENDNOTES

[1] MATLAB is registered trademark of the MathWorks Inc.

[2] ORCAD, CAPTURE, PSPICE and PROBE are copyright by the Cadence Design Systems, Inc.

Figure A.9. The inverter output voltage and the load voltage

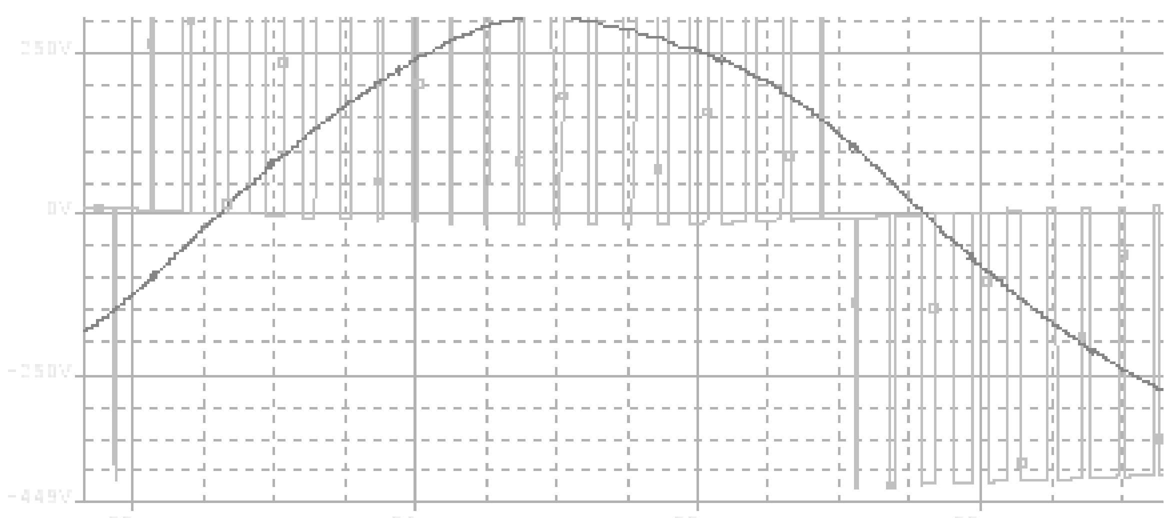


Figure A.10. The harmonic spectrum of the inverter output voltage