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Higher-Order Finite Element Vibration Analysis of Circular Raft on Winkler Foundation

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ABSTRACT

On a Winkler foundation, solid circular plate vibration is examined using a higher-order finite element in polar co-ordinate system. The present formulation has developed a Mat-lab code to handle any boundary conditions. Validation of the code is carried out after the convergence studies. The results are compared to other researchers and show excellent conformity. Furthermore, a parametric analysis gave the first 10 natural frequency characteristics in tabular and graphical form. The authors conclude that the present formulation is straightforward, behaves exceptionally well for thin solid circular plates on elastic foundations with reasonable convergence rate and accuracy, and requires less computational effort, resources, and time.

KEYWORDS

Convergence Studies, Elastic Foundation, Frequency Parameters, Higher-Order Finite Element, Less Computational Effort, Natural Frequency, Solid Circular Plates, Winkler Foundation

INTRODUCTION

Due to their wide range of applications in civil, structural, aeronautical, and mechanical engineering, such as highways, buried pipelines, airport runways, water tanks, and railway tracks, vibration analysis of solid circular plates on elastic foundations has attracted a lot of attention. These constructions provide significant soil-structure interaction issues, and it is difficult to determine how dynamic vertical or horizontal forces are transmitted to the foundation. Therefore, structures of different shapes, materials, and models, such as beams, plates, and shells, are frequently employed. Many engineering problems, including highway pavements, bridges, ships, steel bearing plates on concrete, etc., can be simplified into beams, plates, and shells on elastic foundations. Therefore, a thorough study of solid circular plates on Winkler foundations is required. Although this problem may be solved using most finite element programmes, analytical methods have several advantages for comprehending the

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fundamental concepts of physics and mechanics. The interaction between structures and complex media has been extensively studied, leading to the development of numerous theoretical frameworks for structures with elastic foundations. However, a theoretical approach cannot solve most of these problems, which leads to numerical techniques.

Numerous research has been carried out to predict the plate's response to elastic foundations. A reasonably straightforward model by (Winkler, 1867) is predicated on the notion that each foundation point's reaction forces per unit area correspond to the foundation's deflection. (Celep, 1988; Celep and Turhan, 1990) have published their assessments of circular plates using the Winkler foundation model. Using the Galerkin technique to estimate plate deflection, (Guler and Celep, 1995a; Güler and Guler, 2004) stated a study of a thin circular elastic plate supporting both uniformly distributed and symmetrical stresses on a two-parameter Pasternak foundation. The study also considers the Pasternak foundation's tensionless properties, allowing the plate to lift off the surface.

Circular plates have been investigated by simulating the foundation soil (Galletly, 1959) to create a more accurate foundation model to fix Winkler's shortcomings. (Eisenberger and Clastornik, 1987a, 1987b) presented and compared two methods for resolving the issues of static analysis, vibrations, and stability of beams on unstable two-parameter elastic foundations and a method for resolving the issues of beam buckling and vibrations on a variable Winkler elastic foundation. (Olson and Lindberg, 1970) created two finite plate bending elements in a polar coordinates system. The first element has a sector that is nine degrees of freedom in a circle, whereas the second has a sector that is twelve degrees of freedom in an annulus. A continuum-based model was created by (Elhuni et al., 2019) to forecast the flexural behaviour of an elastic soil layer supporting a circular tank foundation. The authors present the coupling problem for the traditional theory of plates on an elastic axis-symmetric circular foundation by considering the soil-structure system's horizontal and vertical displacement in the polar coordinate system. The issue of vibrations of circular plates resting on a Winkler foundation and elastically constrained against rotation and translation is addressed (Rao and Rao, 2013). (Narita, 1985) investigated free elliptical plates comprehensively using the Ritz approach. The trial functions were performed using power series. There were multiple elliptical plates with different ellipticities, and the first five frequency parameters were presented for each. (Kim and Dickinson, 1989) presented approximate natural frequencies and frequency characteristics for fully clamped and simply supported circular plates for completely free circular plates. (Gupta and Bhardwaj, 2004) investigated how the combination of an elastic foundation and a parabolic thickness variation affected the vibration of elliptical plates. Their study looked at the frequency and mode shapes of the first four vibration modes for different aspect ratios, taper, orthotropic, and foundation parameter values for free plates, simply supported edges and clamped edges.

The governing equations of an elastic circular plate on a tensionless foundation are obtained and numerically solved in (Guler and Celep, 1995b) to investigate the impact of the foundation's tensionless nature on the foundation and the plate's static and dynamic behaviours. On the opening page of (Leissa, 1993)., the vibration of a plate supported laterally by an elastic foundation was explored. Leissa reasoned that a full (Winkler) foundation simply results in a continual increase in the plate's squared natural frequency. (Salari et al., 1987) also predicted this. (Laura et al., 1995) studied the vibration of a plate resting on an elastic foundation, in which a natural frequency connection is no longer valid. (Wang, 2005) studies have various goals. First, they'll calculate exact frequency determinants to validate and extend Laura's approximations for clamped and simply supported plates. Second, they analyze plates with free and moving edges. Past authors' assumptions of a fundamental axisymmetric mode may be erroneous in certain instances. It shall be demonstrated. Using a variational formulation, (Ascione and Grimaldi, 1984) investigated unilateral frictionless contact between a circular plate and a Winkler foundation. Leissa (Leissa, 1993) provided one of the early treatments of this issue by tabulating data for the frequency parameter for four vibration modes of a circular plate that was simply supported and had changing rotational stiffness. A circular plate lying on the Winkler foundation underwent a significant deflection, which (Zheng and Zhou, 1988) examined.

The axisymmetric dynamic response of a circular plate on an elastic foundation was investigated by (Ghosh, 1997). Four nodded sixteen degrees of freedom factor was considered by (Bogner et al., 1966) by adding one additional degree of freedom to every node.

With a higher-order displacement function taken into consideration, the analysis aims to establish an efficient method for determining the static response of a thin circular plate supported by elastic foundations. According to Kirchhoff's theory, a rectangular element needs four degrees of freedom at the right angle corner and at least six degrees at each non-right angle corner for a second-order (C^2) compatible type plate element (Krishnamoorthy, 1987).

The goal of this work is to find a simple method that requires little in the way of computational resources, time, or effort for the analysis of circular plates on elastic foundations.

This paper presents solid circular plates based on Kirchhoff's theory resting on the Winkler foundation for free vibration analysis using a higher-order finite element method in a polar coordinate system. First, a Mat-lab (Inc., 2011) code is written. The first convergence study and validation of the suggested formulation are then conducted, and the findings demonstrate excellent conformity with other studies. Several numerical examples demonstrate the present formulation's convergence rate, precision, and application for free vibration analysis of higher-order circular plates on winkler foundations. The present element's performance, convergence rate, precision, and applicability are all outstanding without dealing with any difficulties.

Four-nodded, twelve-degrees-of-freedom element non-conformity is overcome by the present element. For Kirchhoff plates, this equation is straightforward, converges quickly, and works well. This element creates thin, solid circular and annular plates on an elastic foundation with a few elements. The present formulation reduces computing cost, time, and memory. The only input is the modulus of subgrade reaction to represent the elastic foundation.

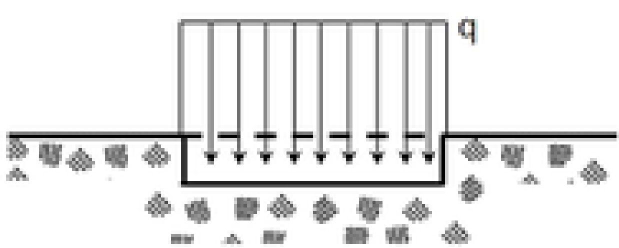
METHODOLOGY

Winkler Model

Winkler introduced a very compact model in the literature, the Winkler foundation (Figure 1). The Winkler model was employed by practising engineers for routine work because of its simplicity. Reaction forces per unit area are assumed to be proportional to foundation deflection in this model. The subgrade reaction modulus, K (Hetenyi, 1950), is a set of equal, independent, discrete, linearly elastic springs that define the vertical deformation properties of the foundation. According to the Winkler model, q = Kw, where w, the vertical soil displacement at that location, is proportional to the contact pressure, and q describes the relationship between external pressure and foundation surface deflection. The soil's "subgrade reaction modulus" is defined as the proportionality constant, K.

A possible representation of the field equation in the domain Ω can be written as

Figure 1. Winkler foundation



$$D\nabla^4 w(r,\theta) + Kw(r,\theta) = q(r,\theta) \tag{1}$$

Where $\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$ is the Laplacian operator and $D = \frac{Eh^3}{12\left(1 - v^2\right)}$ It is defined as the flexural rigidity of the plate. E = the modulus of elasticity of the plate, h = thickness of the plate, ν = Poisson's ratio. q = the applied distributed load, Ω = domain of the plate. Estimating the foundation parameter modulus of subgrade response can be found in Biot (1937), Bowles (1996), Galin (1943), Lenczner (1962), Terzaghi (1955), Vesić (1961), and Dutta et al. (2022, 2021).

Dynamic Equation of Plate on Winkler Foundation

$$D\nabla^4 w(r,\theta) + Kw(r,\theta) = q(r,\theta)$$

is the static equation of the plate on the Winkler foundation.

After applying D'Alembert's principle, the unbalanced force equals the inertia force.

$$\text{Unbalanced force} = q(r,\theta) - \left(D\nabla^4 w(r,\theta) + Kw(r,\theta)\right) \text{and inertia force } m\frac{\partial^2 w}{\partial t^2}$$

$$\therefore\ m\frac{\partial^2 w}{\partial t^2} = q(r,\theta,t) - \left(D\nabla^4 w(r,\theta) + Kw(r,\theta)\right)$$

Considering the damping force $\left(c\frac{\partial w}{\partial t}\right)$ Also, it gets the following

$$D\nabla^4 w(r,\theta) + Kw(r,\theta) + c\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2} = q(r,\theta,t)$$

It is the plate's dynamic equation on the Winkler foundation. m - Mass per unit plate area = ρh, ρ - mass density of plate material and c - Damping constant.

Application of Thin Plate Theory with Annular Sector Element

Displacement variations of thin plate written as (Figure 2).

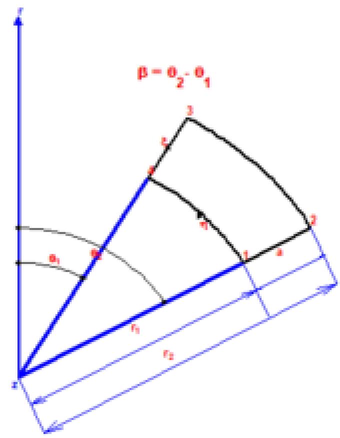
$$u(r,\theta,z) = -z\frac{\partial w}{\partial r};\ v(r,\theta,z) = -\frac{z}{r}\frac{\partial w}{\partial \theta}\ \ and\, w(r,\theta,z) = w(r,\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial\left(-z\frac{\partial w}{\partial r}\right)}{\partial \theta} = -z\frac{\partial^2 w}{\partial r\partial \theta};\frac{\partial v}{\partial \theta} = \frac{\partial\left(-\frac{z}{r}\frac{\partial w}{\partial \theta}\right)}{\partial \theta} = -\frac{z}{r}\frac{\partial^2 w}{\partial \theta^2};\frac{\partial v}{\partial r} = \frac{\partial\left(-\frac{z}{r}\frac{\partial w}{\partial \theta}\right)}{\partial r} = -z\frac{\partial}{\partial r}\left(\frac{\frac{\partial w}{\partial \theta}}{r}\right) = -z\left[\frac{r\frac{\partial^2 w}{\partial r\partial \theta} - \frac{\partial w}{\partial \theta}}{r^2}\right]$$

$$\frac{\partial v}{\partial r} = -\frac{z}{r}\frac{\partial^2 w}{\partial r\partial \theta} + \frac{z}{r^2}\frac{\partial w}{\partial \theta};\mu_{rr} = \frac{\partial u}{\partial r} = -z\frac{\partial^2 w}{\partial r^2};\mu_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{z}{r}\frac{\partial w}{\partial r} - \frac{z}{r^2}\frac{\partial^2 w}{\partial \theta^2};$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = -\frac{z}{r}\frac{\partial^2 w}{\partial r\partial \theta} + \left(-\frac{z}{r}\frac{\partial^2 w}{\partial r\partial \theta} + \frac{z}{r^2}\frac{\partial w}{\partial \theta}\right) + \frac{z}{r^2}\frac{\partial w}{\partial \theta} = -\frac{2z}{r}\frac{\partial^2 w}{\partial r\partial \theta} + \frac{2z}{r^2}\frac{\partial w}{\partial \theta}\ \text{ and } \mu_{zz} = \gamma_{rz} = \gamma_{\theta z} = 0$$

Figure 2. Annular sector element



$$\begin{Bmatrix} \mu_{rr} \\ \mu_{\text{¸¸}} \\ {}^{3}{}_{r\text{¸}} \end{Bmatrix} = \begin{bmatrix} -z\dfrac{\partial^2 w}{\partial r^2} \\ -\dfrac{z}{r}\dfrac{\partial w}{\partial r} - \dfrac{z}{r^2}\dfrac{\partial^2 w}{\partial \text{¸}^2} \\ -\dfrac{2z}{r}\dfrac{\partial^2 w}{\partial r \partial \text{¸}} + \dfrac{2z}{r^2}\dfrac{\partial w}{\partial \text{¸}} \end{bmatrix} = -z\begin{bmatrix} \dfrac{\partial^2 N}{\partial r^2} \\ \dfrac{1}{r}\dfrac{\partial N}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 N}{\partial \text{¸}^2} \\ \dfrac{2}{r}\dfrac{\partial^2 N}{\partial r \partial \text{¸}} - \dfrac{2}{r^2}\dfrac{\partial N}{\partial \text{¸}} \end{bmatrix}\{w\} \therefore \left[B_{bi}\right] = \begin{bmatrix} \dfrac{\partial^2 N}{\partial r^2} \\ \dfrac{1}{r}\dfrac{\partial N}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 N}{\partial \text{¸}^2} \\ \dfrac{2}{r}\dfrac{\partial^2 N}{\partial r \partial \text{¸}} - \dfrac{2}{r^2}\dfrac{\partial N}{\partial \text{¸}} \end{bmatrix}$$

$$\left[B_b\right] = \left[B_{b1} B_{b2} \dots\dots\dots B_{b16}\right]$$

For four nodded elements.

For the plane stress condition, the relationship between stresses and strain is given by,

$$D_{11} = \frac{E}{\left(1 - \text{½}^2\right)} \text{ and } G = \frac{E}{2\left(1 + v\right)};$$

$$D_{22} = D_{11}; D_{33} = D_{11}; D_{12} = vD_{11}; D_{13} = D_{12}; D_{21} = D_{12}; D_{23} = D_{12}; D_{31} = D_{12}; D_{32} = D_{12}; D_{44} = G;$$

$$\therefore \left[C_b\right] = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & D_{44} \end{bmatrix}; \left[D\right] = \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \left[C_b\right] dz = \frac{Eh^3}{12\left(1 - \text{½}^2\right)} \begin{bmatrix} 1 & \text{½} & 0 \\ \text{½} & 1 & 0 \\ 0 & 0 & \dfrac{1-\text{½}}{2} \end{bmatrix}.$$

Where [D] is the plate rigidity matrix.

Characteristic Equation

Using the Hamilton principle (Petyt, 1990), plate-soil equations of motion for free vibration without damping are

$$[M]\left\{\ddot{w}\right\}+[KK]\{w\}=\{0\} \tag{2}$$

Where w is the plate's displacement and $\ddot{w}$ acceleration, respectively, and [KK] and [M] are the stiffness and mass matrices of the plate-soil system. One can get (Hinton, 1988) the natural frequencies and vibrational mode by resolving the generalized Eigenvalue problems.

If a harmonic motion is assumed for free vibration analysis

$$\{\mathrm{w}\}=\{\overline{\mathrm{w}}\}\mathrm{e}^{\mathrm{i\acute{E}t}} \text{ and } \left\{\ddot{\mathrm{w}}\right\}=-\acute{\mathrm{E}}^2\{\overline{\mathrm{w}}\}\mathrm{e}^{\mathrm{i\acute{E}t}}$$

Where $\{\overline{\mathrm{w}}\}$ is the amplitude of $\{\mathrm{w}\}$.

When $\{\mathrm{w}\}$ and $\left\{\ddot{\mathrm{w}}\right\}$ are substituted in equation (2), one obtains

$$\left([\mathrm{K.K.}]-\acute{\mathrm{E}}^2[\mathrm{M}]\right)\{\overline{\mathrm{w}}\}=\{0\} \tag{3}$$

Equation (3)'s non-trivial approach implies that

$\left|\left([\mathrm{K.K.}]-\acute{\mathrm{E}}^2[\mathrm{M}]\right)\right|=0$ the determining equation.' ω' stands for the natural frequency.

Finite Element Formulation

The four nodded annular plate bending (APB) components are depicted in Figure 3. Every node has four degrees of freedom. There are a total of sixteen degrees of freedom for this element. Bogner et al. (1966) designed this compatible element.

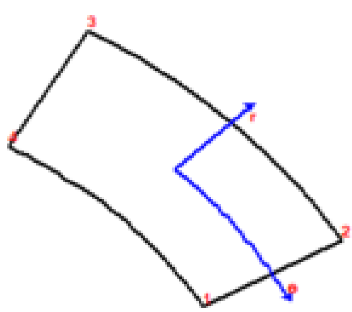
The plate's length and subtended angles are r and β.

Nodes 1, 2, 3, and 4 in polar coordinate systems are $(r_1, 0)$, $(r_2, 0)$, (r_2, β), (r_1, β) respectively. Where $a = r_2 - r_1$.

Considered $w, \frac{\partial \mathrm{w}}{\partial \mathrm{r}}, \frac{\partial \mathrm{w}}{\partial \text{¸}}, \frac{\partial^2 \mathrm{w}}{\partial \mathrm{r}\partial \text{¸}}$ nodal unknown.

The displacements function for this type of element assumed as follows

Figure 3. 4 APB element



$$w(r,\theta) = a_1 + a_2r + a_3\theta + a_4r^2 \ldots\ldots\ldots + a_{16}r^3\theta^3. \quad (4)$$

The shape function can now be calculated from equation (4) by evaluating the sixteen corner displacements. Hence bending stiffness matrix, $\left[K_b\right] = \int_{-1}^{1}\int_{-1}^{1}\left[B_b\right]^T\left[D\right]\left[B_b\right]\left|J\right| drd_{¸}$ and $|J| = ar\beta$.

The bending is described by the usual steps as follows: $\left[K_b\right] = \frac{1}{4}\sum_{j=1}^{4}\sum_{i=1}^{4}W_iW_j\left|J\right|\left[B_b\right]^T\left[D\right]\left[B_b\right]$.

These equations show how bending stiffness can be expressed as a full four by four point Gauss-Legendre quadrature.

The lateral displacement of an area 'dA' normal to the foundation for a structural member with a differential area 'dA' in contact with the foundation is given by w = [N]d. For example, in a linear spring, the strain energy Ur is given by ½Kw². $U_r = \frac{1}{2}\int Kw^2dA = \frac{1}{2}\int K\left\{d\right\}^T\left[N\right]^T\left[N\right]\left\{d\right\}dA$; K is also referred to as the modulus of subgrade reaction. $U_r = \frac{1}{2}\left\{d\right\}^T\left[K_f\right]\left\{d\right\}$. When the element's foundation stiffness matrix is,

$$\left[K_f\right] = \int K\left[N\right]^T\left[N\right]dA = \frac{Kar^2}{4}\int_{-1}^{1}\int_{-1}^{1}\left[N\right]^T\left[N\right]drd_{¸} = \frac{Kar^2}{4}\int_{-1}^{1}\int_{-1}^{1}\left[N\right]^T\left[N\right]drd_{¸} = \frac{K}{4}\int_{-1}^{1}\int_{-1}^{1}\left[N\right]^T\left[N\right]\left|J\right|drd_{¸}\cdot$$

[N] is the same as the shape function matrix of the plate.

For the i[th] node's related foundation parameter, a typical sub-matrix is $\left[K_{fi}\right] = \frac{K}{4}\sum_{i=1}^{4}\sum_{j=1}^{4}W_iW_j\left|J\right|N_i^TN_j$

$$\left[K_f\right] = \left\{K_{f1}\ K_{f2}\ldots\ldots\ldots\ldots K_{f16}\right\}^T$$

for four nodded elements.

Elements′ stiffness matrix is $\left[K\right] = \left[K_b\right] + \left[K_f\right]$.

The global stiffness matrix is generated by using the conventional finite element method. $\therefore \left[K_S\right] = \sum_{1}^{n}\left(\left[K_b\right] + \left[K_f\right]\right)$.

Mass Matrix

Additionally, it is considered that vibration does not cause the subsoil to vary in depth.

The thin plate's kinetic energy is defined by

$$T_e = \frac{1}{2}\acute{A}\int_{A_e}\left(\int_{\frac{-h}{2}}^{\frac{h}{2}}\dot{w}^2dz\right)dA = \frac{1}{2}\acute{A}h\int_{A_e}\dot{w}^2\,dA = \frac{1}{2}\left\{d'\right\}^T\acute{A}h\int_{0}^{1}\left[N\right]^T\left[N\right]dA\left\{d'\right\} \therefore \left[M\right] = \acute{A}h\int_{A_e}\left[N\right]^T\left[N\right]dA$$

$$\therefore \left[M\right] = \acute{A}h\int_{-1}^{1}\int_{-1}^{1}\left[N\right]^T\left[N\right]\left|J\right|drd_{¸}$$

RESULT AND DISCUSSION

This study considers several boundary conditions, such as S.S., C, and F, among others, where S.S. refers to a simply-supported edge, F to a free edge, and C to a clamped edge. Clamped edge, $w = \frac{\partial w}{\partial r} = \frac{\partial w}{\partial_{¸}} = \frac{\partial^2 w}{\partial r\partial_{¸}} = 0$, S.S. edge, $w = \frac{\partial^2 w}{\partial r\partial_{¸}} = 0$ and free edge, $w \neq 0,\ \frac{\partial w}{\partial r} \neq 0,\ \frac{\partial w}{\partial_{¸}} \neq 0,\ \frac{\partial^2 w}{\partial r\partial_{¸}} \neq 0.$

The following dimensionless parameters have been defined to make results comparison simpler.

1. Plate on elastic foundation's natural frequency for nth mode, $\tau_n = \omega_n a^2 \sqrt{\frac{\rho h}{D}}$.
2. Foundation parameter, Kw = Ka^4/D;

Normal modes analysis without loads or constraints will have one translation and two rotation modes. The first three modes have zero or near-zero modal frequency. So, this is a rigid body motion with no vibration. For a free plate, we always use the fourth mode.

Convergence Study

A circular plate (a = 1 m, h = 0.01 m, $\nu = 0.30$ and $\rho = 7850$ kg/m^3) without a foundation is considered first for the convergence study for C, S.S. and F boundary conditions of the present study (P.S.), as well as a convergence study, is performed shown in Figures 4, 5, and 6. The mesh size of 10 × 60, i.e. the number of elements 600, is decided for a good result compared to 15 × 90, i.e. the number of elements 1350 of a 12 DOF element, which requires less time, resources, and computer memory. The results listed in Table 1 and compared with (Lam et al., 1992) show excellent agreement. Shows the proposed formulation's rapid convergence and observed for simply supported and clamped plates converging from the reverse direction.

Table 1. Natural frequency parameter for C, S.S. plate for 1st mode and free (F) plate for 4th mode

Mesh Size		No of Node	C	S.S.	F
			τ_1	τ_1	τ_4
3 ×	20	61	9.5468	4.7219	5.4995
4 ×	24	97	9.7850	4.7975	5.4339
6 ×	36	217	9.9961	4.8641	5.3898
8 ×	40	321	10.0827	4.8916	5.3754
10 ×	60	601	10.1263	4.9055	5.3689
(Lam et al., 1992)			10.2160	4.9352	5.3583
Difference (%)			0.878	0.601	0.197

Figure 4. Frequency parameter vs. no. of node

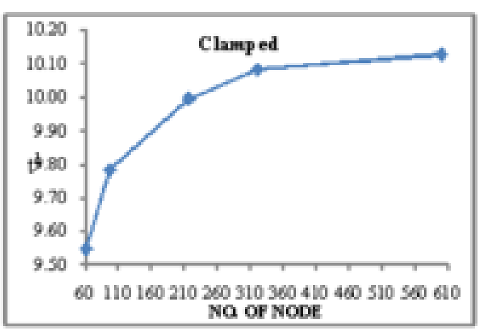


Figure 5. Frequency parameter vs. no. of node

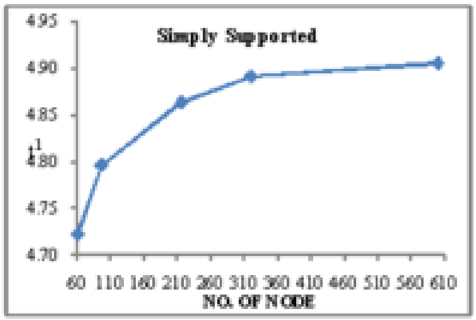
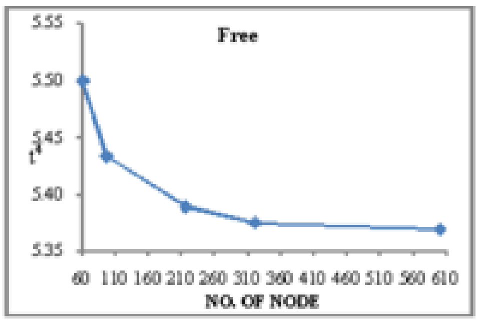


Figure 6. Frequency parameter vs. no. of node



Validation

A circular plate ($\nu = 0.30$) on Winkler foundation (Kw = 200, 500) is considered first to validate the present formulation. The results listed in Table 2, compared with those given in (Sharma and Shivani, 2011), show excellent agreement.

Parametric Study

For parametric study of thin solid circular plate ($\nu = 0.25$) the data has been taken as r/h =100, Kw = 0, 100, 200, 300, 400 and various boundary conditions. The parametric study results are presented in Table 3, and Figures 7–9 show the variation of the first ten frequency parameter. Table 4 shows the impact of plate material Poisson's ratio on the natural frequency parameter.

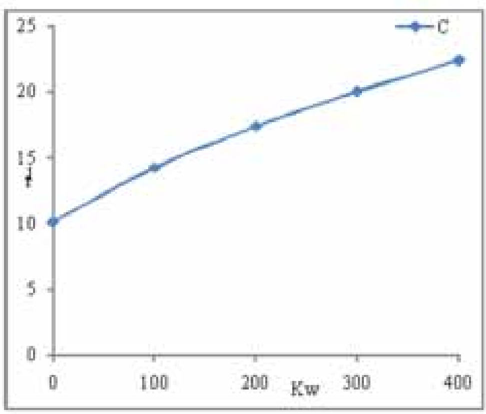
Table 2. Comparison of natural frequency parameter for clamped (C) and simply supported (S.S.) solid circular plate

B.C	Kw	τ_1			τ_2			τ_3		
		(Sharma and Shivani, 2011)	P.S.	Difference (%)	(Sharma and Shivani, 2011)	P.S.	Difference (%)	(Sharma and Shivani, 2011)	P.S.	Difference (%)
C	200	17.4460	17.3938	0.299	42.2107	41.7236	1.154	90.2194	88.8485	1.520
S	200	14.9785	14.9688	0.065	32.9132	32.6286	0.865	75.4925	74.5017	1.312
C	500	24.5838	24.5467	0.151	45.6261	45.1758	0.987	91.8670	90.5210	1.465
S	500	22.8988	22.8925	0.028	37.1925	36.9408	0.677	77.4540	76.4886	1.246

Table 3. Natural frequency parameter of a solid circular plate

B.C.	Kw	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
C	0	10.1287	21.1922	21.1922	34.8237	34.8237	39.2683	50.9303	50.9303	60.5659	60.5659
	100	14.2335	23.4331	23.4331	36.2310	36.2310	40.5216	51.9028	51.9028	61.3859	61.3859
	200	17.3952	25.4776	25.4776	37.5857	37.5857	41.7373	52.8573	52.8573	62.1950	62.1950
	300	20.0647	27.3698	27.3698	38.8933	38.8933	42.9185	53.7950	53.7950	62.9938	62.9938
	400	22.4185	29.1395	29.1395	40.1583	40.1583	44.0681	54.7165	54.7165	63.7826	63.7826
S.S.	0	4.8321	13.8139	13.8139	25.5479	25.5479	29.3576	39.8727	39.8727	48.3007	48.3007
	100	11.1063	17.0535	17.0535	27.4353	27.4353	31.0140	41.1075	41.1075	49.3250	49.3250
	200	14.9449	19.7692	19.7692	29.2009	29.2009	32.5863	42.3064	42.3064	50.3285	50.3285
	300	17.9819	22.1545	22.1545	30.8657	30.8657	34.0862	43.4722	43.4722	51.3123	51.3123
	400	20.5754	24.3069	24.3069	32.4452	32.4452	35.5228	44.6075	44.6075	52.2777	52.2777
F	0	0.0071	0.0041	0.0046	5.5218	5.5218	8.8466	12.7648	12.7648	20.4085	20.4085
	100	10.0000	10.0000	10.0000	11.4232	11.4232	13.3515	16.2155	16.2155	22.7267	22.7267
	200	14.1421	14.1421	14.1421	15.1819	15.1819	16.6812	19.0510	19.0510	24.8295	24.8295
	300	17.3205	17.3205	17.3205	18.1794	18.1794	19.4490	21.5161	21.5161	26.7676	26.7676
	400	20.0000	20.0000	20.0000	20.7483	20.7483	21.8692	23.7264	23.7264	28.5746	28.5746

Figure 7. Frequency parameter vs. Kw for clamped plate



With an increase in sub-grade reaction, this frequency parameter has been observed to rise regardless of boundary conditions. With more edge constraints, this frequency parameter has been observed to rise. Higher edge constraints result in a plate with higher flexural rigidity and, thus, a higher frequency response. Table 4 demonstrates that Poisson's ratio of plate material little impacts the natural frequency parameter.

CONCLUSION

The vibration of thin solid circular plates resting on the Winkler foundation has been studied using the higher-order finite element method in the current polar coordinate system. The process is

Figure 8. Frequency parameter vs. Kw for simply supported plate

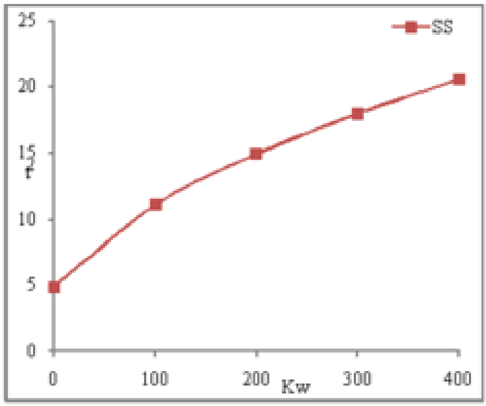


Figure 9. Frequency parameter vs. Kw for free plate

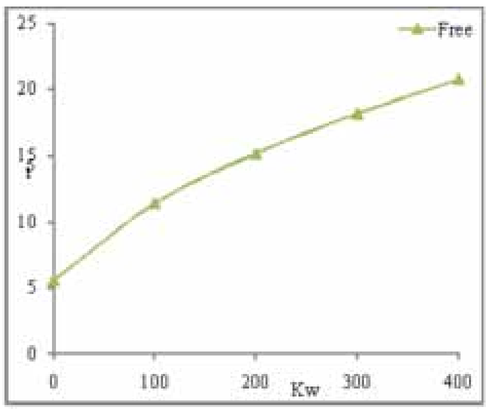


Table 4. Natural frequency parameter of a solid circular plate, Kw = 200

μ	Clamped Plate			Simply Supported Plate		
	τ_1	τ_2	τ_3	τ_1	τ_2	τ_3
0.20	17.3965	41.7502	88.9116	14.9204	32.5430	74.4483
0.25	17.3952	41.7373	88.8812	14.9449	32.5863	74.4761
0.30	17.3938	42.7236	88.8485	14.9688	32.6286	74.5017
0.35	17.3923	41.7090	88.8136	14.9922	32.6698	74.5251

straightforward and can determine frequencies and mode shapes close to the exact ones. Employing the higher-order finite element method in the polar coordinate system, the vibration of thin solid circular plates resting on the Winkler foundation, the first ten eigenvalues are tabulated for different boundary conditions and foundation parameters. The accuracy and efficacy of the proposed formulation for various foundation parameters and boundary conditions are confirmed by the results obtained from other studies. A parametric analysis is also carried out to clarify how different parameters affect the results. The higher-order finite elements for vibration analysis generate a highly accurate approximation solution for thin solid circular plates resting on the Winkler foundation, according to many numerical experiments, including varying support conditions.

The following conclusions are drawn from the numerical results reported in the previous section.

- The approach has a very rapid convergence and produces exceptionally accurate predictions.
- The numerical findings for solid circular plates and various support conditions will not only demonstrsate the usefulness of the present elements. Still, they will also be a handy reference for future researchers in this field and practitioners and design engineers due to their ease of formulation, ability to provide a very accurate approximation solution, cost-effectiveness and less time, resources, and computing effort required.
- The only input is the modulus of subgrade reaction to represent the elastic foundation.
- The findings of this investigation are consistent with accurate and numerical results published in the literature.

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